Sample examination problems

Knowledge is useful only if it can be applied. To see if you can apply your knowledge of calculus, I will pose some examination questions that are different from problems you have done for homework. I hope that the examinations will not only test your understanding but also expand your horizons.

Here are two sample questions that I considered putting on the first examination (but I actually chose other questions instead).

1. The Legendre polynomials are a special sequence of functions that arise in mechanics, electromagnetic theory, and so forth. Some of the properties of the $k$th Legendre polynomial $P_k(x)$ are the following.

   - $P_k(x)$ is a polynomial of degree $k$.
   - $P_k(1) = 1$.
   - $\int_{-1}^{1} [P_k(x)]^2 \, dx = \frac{2}{2k + 1}$.
   - Whenever $k \neq n$, the Legendre polynomials $P_k$ and $P_n$ satisfy the orthogonality relation

$$\int_{-1}^{1} P_k(x) P_n(x) \, dx = 0.$$  

Given that $P_0(x) = 1$ and $P_1(x) = x$, determine the polynomial $P_2(x)$.

2. (a) Give an example of a rational function $r(x)$ such that the improper integral $\int_{0}^{\infty} r(x) \, dx$ converges.

(b) Give an example of a rational function $r(x)$ such that the improper integral $\int_{0}^{\infty} r(x) \, dx$ diverges.

(c) In general, what conditions on a rational function $r$ guarantee that $\int_{0}^{\infty} r(x) \, dx$ converges?