# A sample examination problem on vectors

A curve in three-dimensional space (think of it as the path of a moving particle) may be described in parametric form as $x(t)\mathbf{i} + y(t)\mathbf{j} + z(t)\mathbf{k}$ or as the vector $\mathbf{r}(t)$. [Lines are the special case when $x(t)$, $y(t)$, and $z(t)$ are first-degree polynomials.] The velocity vector $\mathbf{r}'(t) = x'(t)\mathbf{i} + y'(t)\mathbf{j} + z'(t)\mathbf{k}$ is a vector tangent to the curve, and $\mathbf{r}''(t)$ represents acceleration.

The *curvature* $\kappa(t)$ (that symbol is a lowercase Greek letter kappa) may be defined in terms of the vector cross product as follows:

$$\kappa(t) = \frac{|\mathbf{r}'(t) \times \mathbf{r}''(t)|}{|\mathbf{r}'(t)|^3}.$$

1. A circle of radius a in the x–y plane may be described by $\mathbf{r}(t) = a(\cos t)\mathbf{i} + a(\sin t)\mathbf{j} + 0\mathbf{k}$. Show that a circle of radius a has curvature equal to the constant $1/a$.

2. What circle in the x–y plane is tangent at the origin to the parabola $y = x^2$ and also has the same curvature at the origin as the parabola?
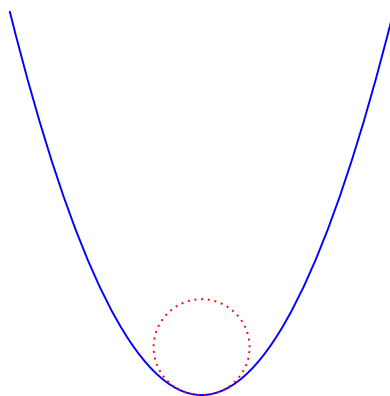
Curves as illustrated in the figure that are tangent and that also have the same curvature at the point of tangency are called *osculating* curves (from the Latin verb *osculari*, to kiss).