Math 152 (honors sections)

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Reminder

Second examination is Wednesday, November 3.

The exam covers through section 10.4.

An old exam is posted at http://www.math.tamu.edu/~boas/courses/152-2002c/exam2.pdf

Convergence tests so far

- If $a_n \not\to 0$, then $\sum_{n=1}^{\infty} a_n$ diverges.
- A geometric series with $|\text{ratio}| < 1$ converges.
- Integral test for positive decreasing functions: the improper integral $\int_1^\infty f(x)\,dx$ and the corresponding series $\sum_{n=1}^{\infty} f(n)$ either both converge or both diverge.
- Inequality comparison test for positive terms: if $0 < a_n < b_n$ (at least for n large) and if $\sum_{n=1}^{\infty} b_n$ converges, then $\sum_{n=1}^{\infty} a_n$ converges too.
- Limit comparison test for positive terms: if $\lim_{n\to\infty} \frac{a_n}{b_n}$ exists (finite limit), and if $\sum_{n=1}^{\infty} b_n$ converges, then $\sum_{n=1}^{\infty} a_n$ converges too.

Root test (not in book)

Example: $\sum_{n=1}^{\infty} \frac{n}{2^n}$ converges

This is not a geometric series, and it is *bigger* than the convergent geometric series $\sum_{n=1}^{\infty} \frac{1}{2^n}$, so the comparison test does not seem to help.

Since $\frac{n}{2^n} = \left(\frac{n^{1/n}}{2}\right)^n$, and since $\lim_{n\to\infty} n^{1/n} = 1$, we have $\frac{n}{2^n} < \left(\frac{1.1}{2}\right)^n$ when n is large, so we can use the comparison test after all: compare to the convergent geometric series $\sum_{n=1}^{\infty} \left(\frac{1.1}{2}\right)^n$.

Root test and ratio test

Root test: If $0 < a_n$, and if $\lim_{n\to\infty} a_n^{1/n} < 1$, then $\sum_{n=1}^{\infty} a_n$ converges. Moreover, if $\lim_{n\to\infty} a_n^{1/n} > 1$, then $\sum_{n=1}^{\infty} a_n$ diverges (because then $a_n \not\to 0$). If $\lim_{n\to\infty} a_n^{1/n} = 1$, the test gives no information.

Ratio test: Exactly the same as the root test, except look at $\lim_{n\to\infty} \frac{a_{n+1}}{a_n}$ instead of $\lim_{n\to\infty} a_n^{1/n}$.

Example: $\sum_{n=1}^{\infty} \frac{3^n (n!)^2}{(2n)!}$.

$$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \frac{3^{n+1}(n+1)!(n+1)!}{(2n+2)!} \Big/ \frac{3^n (n!)(n!)}{(2n)!} = \lim_{n\to\infty} \frac{3(n+1)^2}{(2n+2)(2n+1)} = \frac{3}{4} < 1,$$

so the original series converges.

Series with some negative terms

Negative terms can only help with convergence:
if $\sum_{n=1}^{\infty} |a_n|$ converges, then so does $\sum_{n=1}^{\infty} a_n$.
An absolutely convergent series converges.

Example: $\sum_{n=1}^{\infty} \frac{(-1)^n}{n^2}$ converges because $\sum_{n=1}^{\infty} \frac{1}{n^2}$ converges.
The two series sum to different values, however: $\sum_{n=1}^{\infty} \frac{(-1)^n}{n^2} = \frac{-\pi^2}{12}$, and $\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}$.

Example: $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges (harmonic series),
yet $\sum_{n=1}^{\infty} \frac{(-1)^n}{n}$ converges (in fact to the value $\ln(\frac{1}{2})$).

Alternating series test

If $a_n \downarrow 0$, then $\sum_{n=1}^{\infty} (-1)^n a_n$ converges.
Error estimate: When the alternating series test applies, the sum of the series is trapped between any two consecutive partial sums.

Example: $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n}{n^4} = -1 + \frac{1}{2^4} - \frac{1}{3^4} + \frac{1}{4^4} - \frac{1}{5^4} + \cdots$

$$-1 + \frac{1}{2^4} - \frac{1}{3^4} + \frac{1}{4^4} - \frac{1}{5^4} < \sum_{n=1}^{\infty} \frac{(-1)^n}{n^4} < -1 + \frac{1}{2^4} - \frac{1}{3^4} + \frac{1}{4^4}$$

$$-0.94754 < \sum_{n=1}^{\infty} \frac{(-1)^n}{n^4} < -0.9459$$

Exact value: $\sum_{n=1}^{\infty} \frac{(-1)^n}{n^4} = \frac{-7\pi^4}{720} \approx -0.947033.$

Homework

- Read section 10.4, pages 605–610.
- Do the Suggested Homework problems for section 10.4.

Monday we will review for the exam and look at an old exam.