Calculus

Instructions Please write your solutions on your own paper. Explain your reasoning in complete sentences to maximize credit.

1. Determine a unit vector that has the same direction as the vector $3\vec{i}-4\vec{j}$.

   Solution. The required unit vector can be obtained by dividing the given vector by its length. The length equals $\sqrt{3^2+4^2}$, or 5, so the unit vector is $\frac{3}{5}\vec{i}-\frac{4}{5}\vec{j}$.

2. Determine the vector projection of the vector $\langle 0,1\rangle$ onto the vector $\langle 2,3\rangle$.

   Solution. The vector projection is equal to

   $$\left(\langle 0,1\rangle\cdot\frac{\langle 2,3\rangle}{|\langle 2,3\rangle|}\right)\frac{\langle 2,3\rangle}{|\langle 2,3\rangle|} \qquad \text{or} \qquad \left(\frac{\langle 0,1\rangle\cdot\langle 2,3\rangle}{|\langle 2,3\rangle|^2}\right)\langle 2,3\rangle.$$

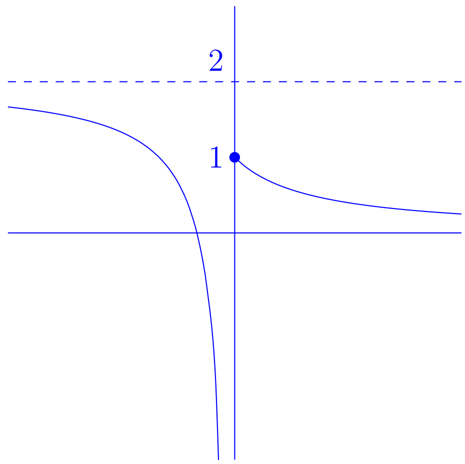
   Since $\langle 0,1\rangle\cdot\langle 2,3\rangle = 0\times 2+1\times 3=3$ and $|\langle 2,3\rangle|^2=2^2+3^2=13$, the vector projection is equal to $\frac{3}{13}\langle 2,3\rangle$, or $\langle\frac{6}{13},\frac{9}{13}\rangle$.

3. Find either a vector equation or a parametric equation for the line that passes through the point with coordinates $(1,7)$ and is parallel to the vector $2\vec{i}+6\vec{j}$.

   Solution. We can get a vector equation by writing a vector that goes from the origin $(0,0)$ to the given point and then adding a multiple of the given direction vector. Thus a vector expression for the line is $\langle 1,7\rangle+t\langle 2,6\rangle$, or $(1+2t)\vec{i}+(7+6t)\vec{j}$, where t takes on all real values. The corresponding Cartesian parametric equations are $x(t)=1+2t$ and $y(t)=7+6t$.

4. Sketch the graph of a function f having the following properties: $\lim_{x\to-\infty} f(x)=2$, $\lim_{x\to\infty} f(x)=0$, $\lim_{x\to 0^-} f(x)=-\infty$, and $\lim_{x\to 0^+} f(x)=1$.

   Solution. The given information indicates that the graph has a horizontal asymptote at height 2 when $x\to-\infty$ and a horizontal asymptote at height 0 as $x\to\infty$. Also, the graph has a vertical asymptote as x approaches 0 from the left-hand side, while the graph approaches height 1 as x approaches 0 from the right-hand side. The figure illustrates such a graph.



Although it was not required to give a formula for such a function, it is not hard to do so. The graph pictured above corresponds to the following function:

$$f(x) = \begin{cases} 2 + \frac{1}{x}, & \text{if } x < 0, \\ \frac{1}{x+1}, & \text{if } x \geq 0. \end{cases}$$

5. Given the information that $f(1) = 2$, $f'(1) = 3$, $g(1) = 4$, and $g'(1) = 5$, determine the value of the derivative $(fg)'(1)$.

   Solution. According to the product rule for derivatives, $(fg)'(1) = f(1)g'(1) + f'(1)g(1) = 2 \times 5 + 3 \times 4 = 22$.

6. If a TI-89 calculator is tossed upward on the moon with an initial velocity of 10 meters per second, then its height in meters after t seconds is equal to $10t - 0.83\,t^2$. Determine the velocity of the calculator after 1 second.

   Solution. The derivative of the height is the velocity, which by the power rule for derivatives equals $10 - 1.66\,t$. When $t = 1$, the velocity equals $10 - 1.66$ or 8.34 meters per second (in the upward direction).

7. (a) State the precise definition of what $\lim_{x \to a} f(x) = L$ means (using ϵ and δ).

(b) Use the precise definition of limit to prove that $\lim_{x \to 2} 5x = 10$.

Solution.

(a) The definition of limit says that to every positive number ϵ there corresponds a positive number δ such that $|f(x) - L| < \epsilon$ whenever $0 < |x - a| < \delta$.

(b) In the specific case at hand, $a = 2$, $f(x) = 5x$, and $L = 10$. I claim that setting δ equal to $\epsilon/5$ will satisfy the definition. Indeed, if $0 < |x - 2| < \epsilon/5$, then $|5x - 10| = 5|x - 2| < 5 \times \epsilon/5 = \epsilon$. Thus the definition is satisfied.

8. The function $2/x$ is never equal to 0 (because a fraction can equal 0 only if the numerator equals 0). This function, however, takes the value -1 when $x = -2$ and the value $+1$ when $x = +2$. Since 0 is a value in between -1 and $+1$, why doesn't this contradict the Intermediate Value Theorem?

Solution. The Intermediate Value Theorem does not apply to the indicated situation, because the function is not continuous on the interval where $-2 \le x \le 2$: namely, the function is not defined when $x = 0$. Since the theorem is not applicable, there is no contradiction.

9. (a) Write the definition of the derivative $f'(x)$ in terms of a limit.

(b) Use the limit definition of the derivative (not the power rule) to show that

$$\frac{d}{dx}\sqrt{x} = \frac{1}{2\sqrt{x}}.$$

Solution.

(a) The definition of the derivative says that

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}.$$

(b) Applying the definition of the derivative, we must compute

$$\lim_{h\to 0} \frac{\sqrt{x+h}-\sqrt{x}}{h}.$$

Multiply the numerator and the denominator both by the expression $\sqrt{x+h}+\sqrt{x}$ to get the equivalent limit

$$\lim_{h\to 0} \frac{(x+h)-x}{h(\sqrt{x+h}+\sqrt{x}\,)} \qquad \text{or} \qquad \lim_{h\to 0} \frac{1}{\sqrt{x+h}+\sqrt{x}}\,.$$

As long as $x > 0$ (this assumption is implicit in the statement of the problem, since the formula does not make sense when $x \le 0$), we can apply the limit laws (or the continuity of the expression with respect to h) to evaluate the final limit by simply setting h equal to 0, yielding the desired result $\frac{1}{2\sqrt{x}}$.

10. **Optional problem for extra credit**

Kim, Lee, and Datta are asked to determine the value of the limit

$$\lim_{x\to -\infty} \frac{2x}{\sqrt{x^2+1}}.$$

Kim argues as follows: "If x has large magnitude, then x^2+1 is nearly equal to x^2, and $\sqrt{x^2} = x$, so the fraction is nearly equal to $2x/x$, and the limit must be 2."

Lee argues as follows: "I remember the trick is to divide the numerator and the denominator both by the highest power of x, which is x^2; then we will have $2x/x^2$ in the numerator, which has limit 0, so the answer must be 0."

Datta argues as follows: "I tried evaluating the function on my calculator. When $x = -10$, I got a function value of -1.99007; when $x = -100$, I got $y = -1.9999$; and when $x = -1000$, I got $y = -2$. When I tried negative values of x of even larger magnitude, I kept getting $y = -2$. Therefore the limit must be -2."

Decide who (if anyone) is correct, and explain the shortcomings in the arguments proposed by these three students.

Solution. When x is negative, the fraction is negative (since the numerator is negative and the denominator is positive), so the limit as $x \to -\infty$ certainly cannot be positive! Therefore Kim is surely wrong. Kim's mistake is that $\sqrt{x^2} = |x|$, not x.

Lee is wrong too, because the term x^2 is inside a square root, so the largest power of x is effectively x to the first power. A correct implementation of Lee's idea could go as follows:

$$\frac{2x}{\sqrt{x^2+1}} = \frac{2x}{\sqrt{x^2}\,\sqrt{1+\frac{1}{x^2}}} = 2 \cdot \frac{x}{|x|} \cdot \left(1 + \frac{1}{x^2}\right)^{-1/2}.$$

Now when $x < 0$, the fraction $x/|x| = -1$. Moreover, when $x \to -\infty$, the term $(1 + x^{-2})^{-1/2} \to 1$. Hence the limit of the whole expression is equal to $2 \cdot (-1) \cdot 1$, or -2.

Thus Datta is correct that the limit equals -2. Although Datta's argument is a convincing numerical demonstration, it does not rise to the level of a proof.