Calculus

Instructions Please write your solutions on your own paper. Explain your reasoning in complete sentences to maximize credit.

1. Suppose vector $\vec{v} = 2\vec{i} + 3\vec{j}$ and vector $\vec{w} = a\vec{i} + b\vec{j}$ for some numbers a and b. If the length $|\vec{w}| = 53$, and the scalar projection of $\vec{w}$ onto $\vec{v}$ equals 45, find the scalar projection of $\vec{w}$ onto $\vec{v}^{\perp}$, the orthogonal complement of $\vec{v}$.
   [Recall that the scalar projection of $\vec{w}$ onto $\vec{v}$ is the component of $\vec{w}$ in the direction of $\vec{v}$, that is, the magnitude of the vector projection of $\vec{w}$ onto $\vec{v}$. In two-dimensional space, the orthogonal complement of a vector is the vector obtained by a 90° counterclockwise rotation.]

2. Recall that the floor function $\lfloor x \rfloor$ is the greatest integer less than or equal to x. (The textbook denotes this function by the symbol $[\![x]\!]$ and calls it the greatest integer function.) Use the Squeeze Theorem to prove that
   $$\lim_{x\to 0}\left(x^2\left\lfloor \frac{1}{x^2}\right\rfloor\right) = 1.$$
   [Neither the TI-89 calculator nor Maple can do this limit!]

3. In February 2006, students at Michigan Tech rolled a world-record snowball, a sphere 6.75 feet in diameter. If the snowball subsequently melted at a rate of 1 cubic foot per day, how fast was the radius of the snowball changing, in units of inches per day, when the radius was equal to 20 inches?
   [One foot equals 12 inches, so one cubic foot equals 1728 cubic inches. The volume of a sphere of radius r equals $\frac{4}{3}\pi r^3$.]

4. The equation $e^y + \ln(x) = 0$ implicitly determines y as a function $f(x)$ when $0 < x < 1$. Show that the graph of f has an inflection point where $x = 1/e$.

5. For how many values of x does the slope of the graph of $\tan^{-1}(x)$ equal the slope of the graph of $1/\tan(x)$? Explain how you know.
   [Recall that $\tan^{-1}(x)$ and $1/\tan(x)$ are different functions, because the exponent -1 means inverse function, not reciprocal.]

6. Determine a number c and a function f such that $3\int_0^x f(t)\,dt = e^{3x} - c$.

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7. Suppose that a particle moves in a circle and has position vector equal to $(\cos t, \sin t)$ at time t. Show that the corresponding velocity vector is orthogonal to the acceleration vector for every t.
   [This is a standard property of circular motion that you may have learned in physics class.]

8. Give a careful statement of **one** of the following definitions:
   (a) the precise definition of limit (using ϵ and δ);
   (b) the definition of derivative (as a limit);
   (c) the definition of the definite integral (as a limit of Riemann sums).

9. Sketch an example of a graph having no points of discontinuity, exactly one critical point, and exactly two inflection points.

10. **Optional problem for extra credit**
    Abby, Blake, and Cory are trying to compute the indefinite integral $\int 2\sin(x)\cos(x)\,dx$.

    Abby says, "Substituting $u = \sin(x)$ and $du = \cos(x)\,dx$, I get

    $$\int 2\sin(x)\cos(x)\,dx = \int 2u\,du = u^2,$$

    so back substituting gives the answer $\sin^2(x)$."

    Blake says, "I remember a trig identity $2\sin(x)\cos(x) = \sin(2x)$, so substituting $u = 2x$ and $du = 2\,dx$, I get

    $$\int 2\sin(x)\cos(x)\,dx = \int \sin(2x)\,dx = \tfrac{1}{2}\int \sin(u)\,du = -\tfrac{1}{2}\cos(u),$$

    and back substituting gives the answer $-\frac{1}{2}\cos(2x)$."

    Cory says, "I tried this problem on my TI-89 calculator, and it gave me the answer $-(\cos(x))^2$."

    Who (if anyone) is right, who is wrong, and what mistakes did these students make?