Calculus

Instructions Please write your name in the upper right-hand corner of the page. Write complete sentences to explain your solutions.

1. Suppose $f(x) = \dfrac{\cos x}{2 + \sin x}$. Find the absolute maximum value of this function for x in the closed interval $[0, 2\pi]$.
   [This is exercise 50 on page 313 of the textbook.]

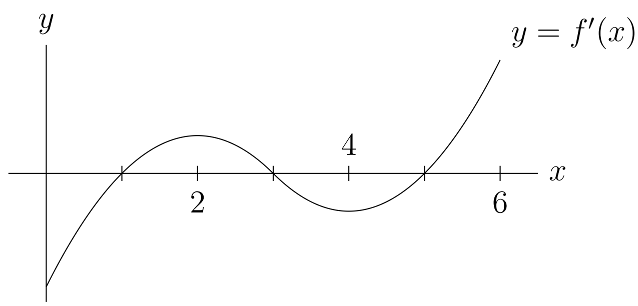
   Solution. By the quotient rule,

   $$f'(x) = \frac{(2+\sin x)(-\sin x) - (\cos x)(\cos x)}{(2+\sin x)^2} = \frac{-2\sin x - \sin^2 x - \cos^2 x}{(2+\sin x)^2}.$$

   Since $\sin^2 x + \cos^2 x = 1$, the derivative $f'(x)$ is equal to 0 if and only if $\sin x = -1/2$, which means that $\cos x = \pm\sqrt{3}/2$. Substituting these values for $\sin x$ and $\cos x$ back into the expression for $f(x)$ shows that at the critical points, the value of the function is $\pm(\sqrt{3}/2)/(3/2)$, or $\pm\sqrt{3}/3$, or $\pm 1/\sqrt{3}$.

   At the endpoints 0 and 2π, the function has the value $1/2$. Since $1/2 < 1/\sqrt{3}$, the maximum value of the function is $1/\sqrt{3}$ (taken when $x = 11\pi/6$).

2. The graph below shows the *derivative* $f'(x)$ on the open interval $(0, 6)$. Determine the values of x for which the graph of the *original function* $f(x)$ [not shown] has (a) local minima and (b) inflection points.



   Solution. (a) Local minima occur when $f(x)$ changes from decreasing to increasing, or when $f'(x)$ changes from negative to positive, that is, at $x = 1$ and $x = 5$. (At $x = 3$, there is a local maximum, because the derivative changes from positive to negative.)

(b) There are inflection points when the derivative changes from increasing to decreasing (or vice versa). This happens when $x = 2$ and $x = 4$.

3. Suppose f is a function that has derivatives of all orders. If $f(0) = 0$ and $f'(0) = 2$, compute the limit $\lim_{x\to 0} \dfrac{f(x)\sin(3x)}{1 - e^{x^2}}$.

Solution. *Method 1.* This is a $0/0$ indeterminate form, so by l'Hospital's rule, the limit equals

$$\lim_{x\to 0} \frac{f'(x)\sin(3x) + 3f(x)\cos(3x)}{-2xe^{x^2}}.$$

The new limit is still an indeterminate form, and another application of l'Hospital's rule gives

$$\lim_{x\to 0} \frac{f''(x)\sin(3x) + 3f'(x)\cos(3x) + 3f'(x)\cos(3x) - 9f(x)\sin(3x)}{-2e^{x^2} - 4x^2e^{x^2}}.$$

Substituting $x = 0$ gives the result -6.

Method 2. A little trickery will reduce the computational complexity. We know that $\lim_{x\to 0} \frac{\sin(3x)}{3x} = 1$, and the given information implies that $\lim_{x\to 0} \frac{f(x)}{2x} = 1$. Therefore

$$\lim_{x\to 0} \frac{f(x)\sin(3x)}{1 - e^{x^2}} = \lim_{x\to 0} \frac{6x^2}{1 - e^{x^2}} = \lim_{t\to 0^+} \frac{6t}{1 - e^t}.$$

Now a single application of l'Hospital's rule produces the answer -6.

4. Sketch the graph of a function f that satisfies all of the following conditions.

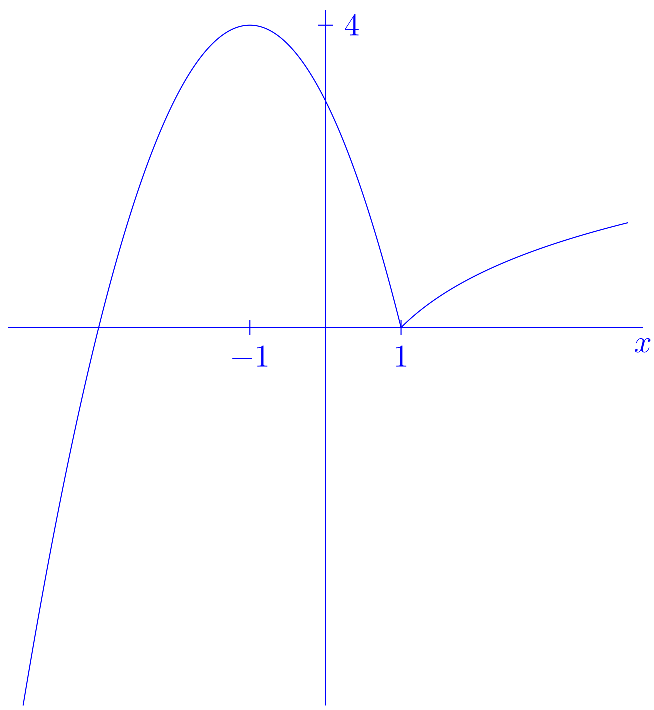
- Conditions on the function: $f(-1) = 4$ and $f(1) = 0$.
- Conditions on the derivative: $f'(-1) = 0$ and $f'(1)$ does not exist.
- Additional conditions on the derivative: $f'(x) < 0$ if $|x| < 1$ and $f'(x) > 0$ if $|x| > 1$.

Calculus

- Condition on the second derivative: $f''(x) < 0$ if $x \neq 1$.

[This is exercise 16 on page 306 of the textbook.]

Solution. On $(-\infty, -1)$, the graph of the function f is increasing and concave down; on $(-1, 1)$, the graph is decreasing and concave down; there is a local maximum at $x = -1$; and on $(1, \infty)$, the graph is increasing and concave down. Since $f'(1)$ does not exist, either the left-hand derivative at $x = 1$ and the right-hand derivative are unequal, or one of these one-sided derivatives fails to exist. An example of a graph satisfying all these properties is shown below.



This particular graph corresponds to the following piecewise-defined function:

$$f(x) = \begin{cases} 4 - (x+1)^2, & \text{if } x \leq 1, \\ \ln(x), & \text{if } x > 1. \end{cases}$$