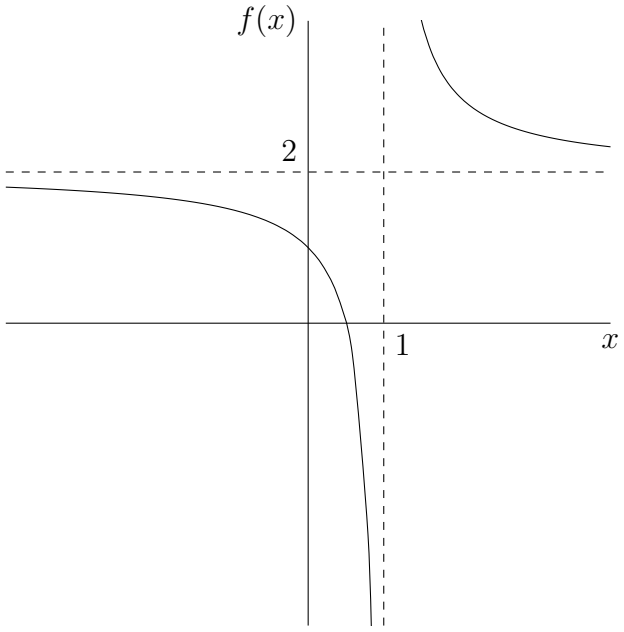
Calculus

Instructions Please write your name in the upper right-hand corner of the page. Write complete sentences to explain your solutions.

1. Write down a function $f(x)$ whose graph looks like the picture. The key features of the picture are that $\lim_{x\to 1^+} f(x) = \infty$, $\lim_{x\to 1^-} f(x) = -\infty$, $\lim_{x\to\infty} f(x) = 2$, and $\lim_{x\to-\infty} f(x) = 2$. Explain the reasoning for your choice of $f(x)$.



Solution. One way to get a suitable function is to start with the simplest function you know that has both horizontal and vertical asymptotes and then adjust the function.

The function $1/x$ has the coordinate axes as asymptotes. Shifting the function to $1/(x-1)$ adjusts the vertical asymptote to the required place. Further shifting the function to $2 + \dfrac{1}{x-1}$ gives the required horizontal asymptote. You could simply the expression to $\dfrac{2x-1}{x-1}$.

There are other functions that have the same asymptotes, but this one is the simplest that matches the x and y intercepts.

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2. The TI-89 calculator says that $\lim_{x\to 1}\left(\frac{1}{x-1}-\frac{2}{x^2-1}\right)=\frac{1}{2}$. Supply a computation that confirms this value. (Suggestion: combine the fractions with a common denominator and simplify.)

Solution. When $x \neq 1$, we can simplify the expression algebraically as follows:

$$\frac{1}{x-1}-\frac{2}{x^2-1}=\frac{x-1}{x^2-1}=\frac{1}{x+1}.$$

The expression on the right-hand side is a rational function whose denominator is not equal to 0 when $x = 1$, so the function is continuous at $x = 1$. Therefore we can correctly obtain the limit by substituting in the value of x: namely,

$$\lim_{x\to 1}\left(\frac{1}{x-1}-\frac{2}{x^2-1}\right)=\lim_{x\to 1}\frac{1}{x+1}=\frac{1}{2}.$$

3. Find a number c such that $\lim_{x\to\infty}\left(\sqrt{x^2+cx}-x\right)=3$.

Solution. We make some algebraic manipulations to convert the expression into a more manageable form. Multiplying and dividing by $\sqrt{x^2+cx}+x$ gives the equivalent expression

$$\frac{x^2+cx-x^2}{\sqrt{x^2+cx}+x}=\frac{c}{\sqrt{1+\frac{c}{x}}+1}.$$

Now $\lim_{x\to\infty}\frac{c}{x}=0$, so the limit of the whole expression is equal to $\frac{c}{\sqrt{1+0}+1}$ or $c/2$. In order for this result to equal 3, we must have $c = 6$.