Calculus

Instructions Please write your name in the upper right-hand corner of the page. Write complete sentences to explain your solutions.

1. Use the definition of the derivative as a limit to prove that $\dfrac{d}{dx}x^2 = 2x$.

   Solution. According to the definition of derivative,
   $$f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h}.$$
   Substituting x^2 for $f(x)$ in this definition, we get
   $$\lim_{h\to 0} \frac{(x+h)^2 - x^2}{h},$$
   which simplifies algebraically to
   $$\lim_{h\to 0} \frac{2xh + h^2}{h}.$$
   The factor of h in the denominator divides out, leaving $\lim_{h\to 0}(2x+h)$, which evidently equals $2x$.

2. Find an equation for the line tangent to the curve $y = x^3$ at the point on the curve where $x = 2$.
   [We now officially know the power rule for derivatives, and you may use that rule in answering this question.]

   Solution. Since $dy/dx = 3x^2$ by the power rule for derivatives, the slope of the curve at the point where $x = 2$ is equal to 12. The point on the curve has coordinates $(2, 8)$, since $y = 8$ when $x = 2$. The point-slope formula implies that the tangent line has the equation $y - 8 = 12(x - 2)$. This result could also be written as $y = 12x - 16$.

3. Give an example of a function $f(x)$ such that the derivative $f'(0)$ does not exist.
   [There are many possible correct answers. Give a one-sentence explanation of why your example works.]

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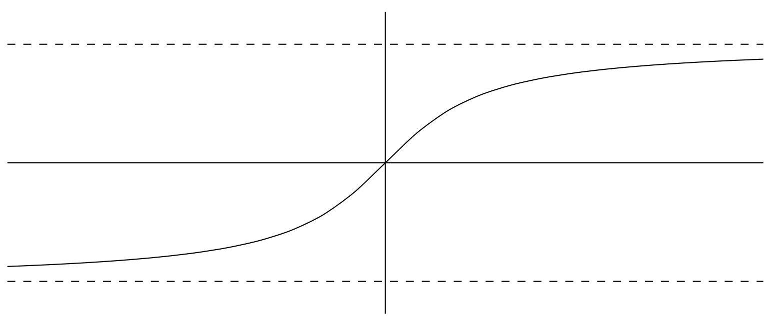
Solution. One solution is a function like $\lfloor x \rfloor$ (the floor function or greatest integer function) that is discontinuous at 0, since a differentiable function has to be continuous.

Another solution is a function like $|x|$ that has a corner in its graph. This function has a right-hand derivative at 0 equal to $+1$ and a left-hand derivative at 0 equal to -1. Since the one-sided derivatives do not match, the derivative does not exist at the point where $x = 0$.

Yet another solution is a function like $x^{1/3}$ that has a vertical tangent line when $x = 0$.

There are many other examples. For instance, we looked in class at the function that equals $x \sin(1/x)$ when $x \neq 0$ and equals 0 when $x = 0$. This function is continuous (by the squeeze theorem) but does not even have one-sided derivatives at the point where $x = 0$.

4. The figure shows the graph of a certain function. Draw a sketch of the graph of the derivative of this function.



Solution. The key features of the derivative (that is, the slope of the illustrated curve) are that it is always positive, it tends to 0 when $x \to \pm\infty$, and it is symmetric about the vertical axis (since the graph shown above is anti-symmetric). Also it increases to a maximum value at the point where $x = 0$, and it decreases thereafter. Here is a graph with these properties: