Review: Vectors

- length
- dot product
- projection
- unit vector
- parametric equations

Squeeze theorem: A tricky example

$$\lim_{x \to \infty} \frac{\sqrt{3x^3 + (\ln x)^2}}{x^2 + (\cos x)^2} = ?$$

Since $x \to \infty$, we can assume $x > 1$. Then $(\ln x)^2 < x^2 < x^3$ so

$$0 \le \frac{\sqrt{3x^3 + (\ln x)^2}}{x^2 + (\cos x)^2} \le \frac{\sqrt{3x^3 + x^3}}{x^2 + (\cos x)^2} \le \frac{2x^{3/2}}{x^2 + 0} = \frac{2}{x^{1/2}} \xrightarrow{x \to \infty} 0$$

The original fraction is trapped between 0 and an expression that has limit 0, so the squeeze theorem implies that the original limit equals 0.

The precise definition of limit: Example

Problem

Prove using the definition of limit that $\lim_{x \to 1} \sqrt{x+3} = 2$.

The definition

For every positive number ε, there exists a positive number δ such that $\left|\sqrt{x+3} - 2\right| < \varepsilon$ whenever $|x-1| < \delta$.

Solution

Multiply and divide by $\sqrt{x+3} + 2$ to see that

$$\left|\sqrt{x+3} - 2\right| = \left|\frac{(x+3) - 2^2}{\sqrt{x+3} + 2}\right| \leq \frac{|x-1|}{0+2}$$

This expression is less than ε when $|x-1| < 2\varepsilon$. So if $\delta = 2\varepsilon$, then the inequality $|x-1| < \delta$ guarantees that $\left|\sqrt{x+3} - 2\right| < \varepsilon$.