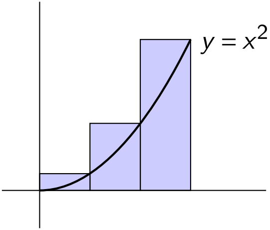
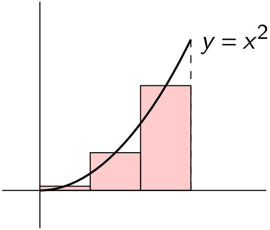
Reminders

- Our last class meeting is Thursday, April 25
  (because Tuesday, April 30 is redefined as Friday).

- The comprehensive final exam takes place 3:00–5:00 in the afternoon of Thursday, May 2.

Recap from last week





If the limit of Riemann sums $\sum_{i=1}^{n} f(x_i^*)\Delta x$ exists as $n \to \infty$, then the limit represents the (signed) area between the graph and the horizontal axis, written $\int_a^b f(x)\,dx$ and called the definite integral.

Some special cases

$$\int_5^5 e^{x^3} \arctan(x^2)\, dx = 0$$ (there is no interval, so zero area).

$$\int_3^{-2} \sin(x)\, dx = -\int_{-2}^{3} \sin(x)\, dx$$
(by convention when the "limits of integration" are reversed).

$$\int_{-5}^{5} x^2\, dx = 2\int_0^5 x^2\, dx$$
by symmetry, since the integrand is an even function.

$$\int_{-5}^{5} x^3\, dx = 0$$
since the integrand is an odd function.

A remark on notation

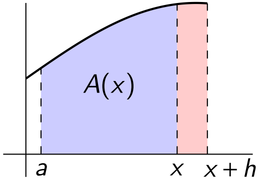
The letter x in $\int_a^b f(x)\,dx$ could be replaced by any other letter:
$\int_a^b f(t)\,dt$ means the same thing.

And so do $\int_a^b f(\xi)\,d\xi$ and $\int_a^b f(\wp)\,d\wp$ and $\int_a^b f(\aleph)\,d\aleph$.

The integration variable is a placeholder,
a so-called "dummy variable."

The fundamental theorem of calculus

What is the derivative of the area function?



$A(x+h)-A(x)$ is approximately the area of a rectangle of width h and height equal to the height of the graph at x, so

$A'(x) = \lim_{h\to 0} \dfrac{A(x+h)-A(x)}{h}$ = the height of the graph at x

(assuming that the graph represents a continuous function).

The fundamental theorem, in symbols

$$\frac{d}{dx}\int_a^x f(t)\,dt = f(x)$$

Example

$$\frac{d}{dx}\int_1^x \sin(t^3)\,dt = \sin(x^3)$$

$$\frac{d}{dx}\int_0^{x^2} e^t\,dt = 2x\,e^{x^2} \text{ by the chain rule}$$

$$\frac{d}{dx}\int_x^5 \cos(t)\,dt = \frac{d}{dx}\left(\int_{-100}^5 \cos(t)\,dt - \int_{-100}^x \cos(t)\,dt\right) = -\cos(x)$$

Second part of the fundamental theorem

Since the derivative of $\int_a^x f(t)\,dt$ equals $f(x)$,

the expression $\int_a^x f(t)\,dt$ is an antiderivative of $f(x)$.

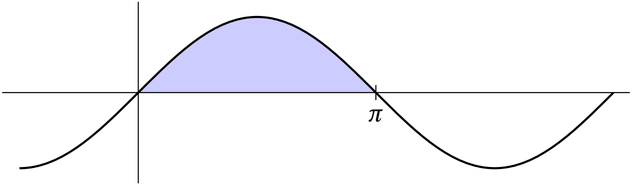
So $\int_a^x f(t)\,dt = F(x) + C$, and we can determine the value of the constant C by setting x equal to a.

Indeed, $\int_a^a = 0 = F(a) + C$, so $C = -F(a)$.

Thus $\int_a^x f(t)\,dt = F(x) - F(a)$.

Example

Find the area under one arch of the sine graph.



Solution: An antiderivative of $\sin(x)$ is $-\cos(x)$, so the area equals

$$\int_0^{\pi} \sin(t)\,dt = -\cos(\pi) - (-\cos(0)) = 2$$

Exercises for this week (not to hand in)

- Section 5.4: Exercises 7, 9, 11, 21, 23, 25, 27, 29, 33, 45, 49.
- Section 5.5: Exercises 1, 3, 5, 7, 11, 13, 17, 21, 25, 27, 33, 53, 59, 67, 69.