Quiz solutions

January 31, 2019

1. When is the first exam?

   Solution. The first exam takes place in class on Thursday, February 14.

2. Compute $\lim_{x\to 1} \frac{\sqrt{x}-1}{x-1}$.

   Solution. Multiply the numerator and the denominator by $\sqrt{x}+1$:

$$\frac{\sqrt{x}-1}{x-1} = \frac{(\sqrt{x}-1)(\sqrt{x}+1)}{(x-1)(\sqrt{x}+1)} = \frac{x-1}{(x-1)(\sqrt{x}+1)} = \frac{1}{\sqrt{x}+1}$$

   (as long as $x \neq 1$). Therefore

$$\lim_{x\to 1} \frac{\sqrt{x}-1}{x-1} = \lim_{x\to 1} \frac{1}{\sqrt{x}+1} = \frac{1}{2}.$$

3. $\lim_{x\to 2} \frac{x^2-4x+4}{|x-2|} = ?$

   Which of the following is the best answer, and why?

   (a) $+\infty$ (b) $-\infty$ (c) 0 (d) does not exist

   Solution. The answer is 0. To see why, notice that the numerator x^2-4x+4 factors as $(x-2)^2$, so the difficulty is mainly that there is an absolute value in the denominator.

   Method 1. Observe that $(x-2)^2 = |x-2|^2$, since the square of a real number is positive (or zero). Therefore

$$\lim_{x\to 2} \frac{x^2-4x+4}{|x-2|} = \lim_{x\to 2} \frac{|x-2|^2}{|x-2|} = \lim_{x\to 2} |x-2| = 0.$$

Method 2. Eliminate the absolute value by considering one-sided limits. If $x > 2$, then $|x-2| = x-2$, so

$$\lim_{x\to 2^+} \frac{x^2-4x+4}{|x-2|} = \lim_{x\to 2^+} \frac{(x-2)^2}{x-2} = \lim_{x\to 2^+}(x-2) = 0.$$

And if $x < 2$, then $|x-2| = 2-x$, so

$$\lim_{x\to 2^-} \frac{x^2-4x+4}{|x-2|} = \lim_{x\to 2^-} \frac{(x-2)^2}{2-x} = \lim_{x\to 2^-} \frac{(2-x)^2}{2-x} = \lim_{x\to 2^-}(2-x) = 0.$$

Since the one-sided limits match up, the two-sided limit $\lim_{x\to 2} \dfrac{x^2-4x+4}{|x-2|}$ exists and equals 0.

Method 3. Express $\dfrac{x^2-4x+4}{|x-2|}$ as $\dfrac{(x-2)}{|x-2|}(x-2)$ when $x \neq 2$. Now

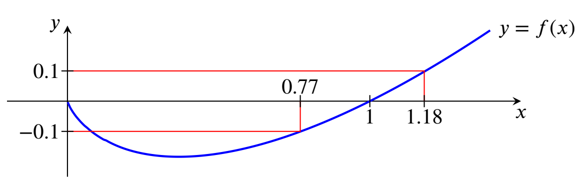
$$-1 \leq \frac{(x-2)}{|x-2|} \leq 1,$$

and $\lim_{x\to 2}(x-2) = 0$, so the Squeeze Theorem applies and shows that

$$\lim_{x\to 2} \frac{(x-2)}{|x-2|}(x-2) = 0.$$

4. Use the graph of f below to find a positive number δ such that

$$\text{if} \quad |x-1| < \delta, \quad \text{then } |f(x)| < 0.1.$$



Solution. The graph indicates that if $0.77 < x < 1.18$, then $|f(x)| < 0.1$. The restriction on x can be rewritten as the statement that $-0.23 < x-1 < 0.18$. Any smaller range of values for x also will guarantee that $|f(x)| < 0.1$. Since the goal is to restrict x by a *symmetric* inequality of the form $-\delta < x-1 < \delta$, the optimal choice of δ is 0.18. Any smaller positive value for δ will work too.