# Final Examination

**Instructions.** Your solution to each problem should include at least one complete sentence. When making a computation, please state your strategy. (For example: "Now I calculate the first derivative by applying the quotient rule.")

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1. Which of the following three numbers is smallest? Explain how you know.

   (a) $|\langle 1, -2\rangle|$ (b) $3\vec{\imath}\cdot(4\vec{\imath}-5\vec{\jmath})$ (c) $\displaystyle\lim_{x\to 0}\frac{e^{6x}-1}{7x}$

   **Solution.** Simply compute all three quantities and compare the values.

   (a) The length $|\langle 1, -2\rangle|$ equals $\sqrt{1^2+(-2)^2}$, or $\sqrt{5}$.

   (b) Since $\vec{\imath}\cdot\vec{\imath}=1$, and $\vec{\imath}\cdot\vec{\jmath}=0$, the indicated dot product equals 12.

   (c) By l'Hôpital's rule and the chain rule, the limit equals $\displaystyle\lim_{x\to 0}\frac{6e^{6x}}{7}$, or $\dfrac{6}{7}$.

   Evidently the smallest of the three numbers is the last one, $\dfrac{6}{7}$.

2. Suppose the position vector $\vec{r}(t)$ of a curve is $\ln(t)\vec{\imath}+\sin(\pi t)\vec{\jmath}$ when $t>0$. Find an equation of the line tangent to the curve at the point where $t=1$.

   **Solution.** Since $\dfrac{d\vec{r}}{dt}=\dfrac{1}{t}\vec{\imath}+\pi\cos(\pi t)\vec{\jmath}$, the tangent vector at the indicated point is $\vec{\imath}+\pi\cos(\pi)\vec{\jmath}$, or $\vec{\imath}-\pi\vec{\jmath}$. The corresponding slope is $-\pi$.

   Now $\vec{r}(1)=\ln(1)\vec{\imath}+\sin(\pi)\vec{\jmath}=0\vec{\imath}+0\vec{\jmath}$, so the point on the curve is the origin. By the point-slope formula, an equation of the tangent line at this point is $y=-\pi x$.

3. Consider the slope $\dfrac{dy}{dx}$ at the point on the graph where $x=0$. For which of the following equations is that slope largest? Explain how you know.

   (a) $y=\dfrac{1+x}{1-x}$ (b) $y=x\tan(x)$ (c) $x^2+xy+y^3=1$

   **Solution.** Compute the three slopes as follows.

   (a) By the quotient rule, $\dfrac{dy}{dx}=\dfrac{(1-x)(1)-(1+x)(-1)}{(1-x)^2}=\dfrac{2}{(1-x)^2}$, so the slope when $x=0$ is equal to 2.

   (b) By the product rule, $\dfrac{dy}{dx}=x\cdot(\sec x)^2+1\cdot\tan x$, so the slope when $x=0$ is equal to 0.

(c) Observe first that if $x = 0$, then $y^3 = 1$, so $y = 1$. Thus the point on the graph is the point $(0, 1)$.

To find the slope, use the method of implicit differentiation (and the product rule and the chain rule) to say that

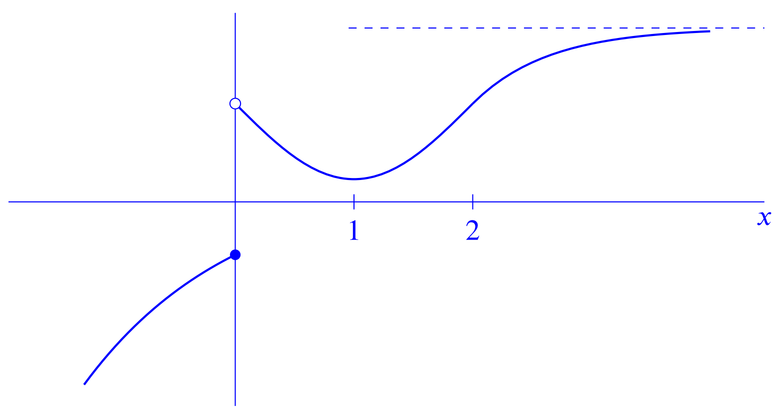
$$2x + x\frac{dy}{dx} + 1 \cdot y + 3y^2\frac{dy}{dx} = 0.$$

When $x = 0$ and $y = 1$, this equation reduces to saying that $1 + 3\frac{dy}{dx} = 0$, so the slope is equal to $-\frac{1}{3}$.

Evidently equation (a) has the largest slope.

4. Sketch the graph of a function having all of the following properties: the first derivative is positive when $x < 0$; the second derivative is negative when $x < 0$; the function has a discontinuity when $x = 0$; there is a local minimum when $x = 1$; there is an inflection point when $x = 2$; and there is a horizontal asymptote when $x \to +\infty$.

**Solution.** The graph must be increasing and concave down when $x$ is negative. The simplest way to produce a discontinuity when $x = 0$ is to have the graph jump there. The concavity changes when $x = 2$. The figure indicates one of many ways to fit all the information together.




5. Which of the following integrals is largest? Explain how you know.

(a) $\displaystyle\int_0^1 x^2\,dx$ (b) $\displaystyle\int_0^1 \sqrt{x}\,dx$ (c) $\displaystyle\int_0^1 \frac{x}{(1+x^2)^2}\,dx$

**Solution.** *Method 1.* The integrands are positive, so each integral represents an area under a curve. If one curve is higher than another, then the corresponding integral is larger.

When $0 < x < 1$, squaring $x$ gives a *smaller* value, and taking the square root of $x$ gives a *larger* value. (For example, $(1/4)^2 = 1/16 < 1/4$, and $\sqrt{1/4} = 1/2 > 1/4$.) Therefore $x^2 < x < \sqrt{x}$ when $0 < x < 1$, so integral (b) has a larger value than integral (a). Notice too that $(1 + x^2)^2$ is larger than 1 when $0 < x < 1$, so $\dfrac{x}{(1 + x^2)^2} < x < \sqrt{x}$. Therefore integral (b) has a larger value than integral (c).

The conclusion is that integral (b) is the largest of the three integrals.

*Method 2.* Compute the integrals as follows.

(a) By the power rule, $\displaystyle\int_0^1 x^2\,dx = \frac{1}{3}x^3 \Big|_0^1 = \frac{1}{3}$.

(b) By the power rule, $\displaystyle\int_0^1 \sqrt{x}\,dx = \int_0^1 x^{1/2}\,dx = \frac{2}{3}x^{3/2} \Big|_0^1 = \frac{2}{3}$.

(c) Make a substitution. If $u = 1 + x^2$, then $\dfrac{du}{dx} = 2x$, so $\dfrac{1}{2}\,du = x\,dx$. Therefore

$$\int_0^1 \frac{x}{(1 + x^2)^2}\,dx = \int_{u(0)}^{u(1)} \frac{1}{u^2}\,\frac{1}{2}\,du = -\frac{1}{2}\cdot\frac{1}{u}\Big|_{u(0)}^{u(1)} = -\frac{1}{2}\cdot\frac{1}{1+x^2}\Big|_0^1 = \frac{1}{4}.$$

Comparing the three results reveals that integral (b) is largest.

6. Is there a positive value of $x$ for which $x + \cos(x) = 0$ ? Explain why or why not.

**Solution.** *Method 1.* The derivative of the function equals $1 - \sin(x)$, and $\sin(x) \le 1$, so $1 - \sin(x)$ is never negative. Therefore the original function never decreases when $x$ increases. When $x = 0$, the function has the value $0 + \cos(0)$, or 1. Since the function never decreases, the function stays at least as large as 1 when $x$ is positive. Thus there is no positive value of $x$ for which $x + \cos(x) = 0$ .

*Method 2.* If $x$ is a positive number for which $\cos(x) \ge 0$, then $x + \cos(x) \ge x + 0 > 0$. And if $x$ is a positive number for which $\cos(x) < 0$, then $x$ must be greater than $\frac{\pi}{2}$, which means that $x + \cos(x) > \frac{\pi}{2} + \cos(x) \ge \frac{\pi}{2} - 1 > 0$ (since $\pi > 2$). Putting the two cases together reveals that $x + \cos(x)$ is always positive when $x$ is positive. Thus there is no positive value of $x$ for which $x + \cos(x) = 0$ .

7. State *two* of the following three theorems.

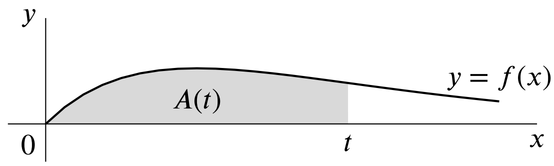
   (a) the squeeze theorem for limits

(b) the intermediate-value theorem

(c) the mean-value theorem

**Solution.** See the textbook for the statements.

8. *Optional extra-credit problem.* Suppose $f(x) = xe^{-x}$, and let $A(t)$ denote the area under the graph of $f$ between $x = 0$ and $x = t$, as indicated in the diagram. Determine $\dfrac{dA}{dt}$, the rate of change of the area, at the value of $t$ for which $f(t)$ is maximal.




**Solution.** The statement presupposes that there *is* a positive value of $t$ for which the continuous function $f(t)$ attains a maximum. Since the extreme-value theorem does not apply on unbounded intervals, there is something to check. Notice that $f(x)$ is positive when $x$ is positive, and $f(0) = 0$, and $\lim_{x\to+\infty} f(x) = 0$. (To confirm the value of this limit, observe that $xe^{-x} = x/e^x$, and apply l'Hôpital's rule.) So if $b$ is sufficiently large, then applying the extreme-value theorem on the interval $[0, b]$ shows that the function $f$ does attain a maximum at some positive number.

Now to solve the problem, observe that the rate of change of the area is equal to the height of the graph (by the fundamental theorem of calculus). So the problem asks for the height of the graph at the point where the height is maximal. In other words, the task is simply to determine the maximal height of the graph.

By the product rule and the chain rule, $f'(x) = x \cdot e^{-x}(-1) + 1 \cdot e^{-x} = e^{-x}(1 - x)$. The exponential function is never equal to 0, so $f'(x) = 0$ if and only if $x = 1$. This critical number is the only candidate for a local maximum of $f$.

Since there is only a single critical number, and the function $f$ does attain an absolute maximum at some positive number, the absolute maximum must occur at the critical number. The answer to the problem is therefore $f(1)$, that is, $e^{-1}$.