Calculus

Math 172-502 First Examination Fall 2005

Work all five problems. These are essay questions. To obtain maximal credit, show your work and explain your reasoning.

1. Show that the area of the region enclosed by the two curves $y = x^2 - 4x$ and $y = 2x$ equals 36. (The curves intersect at $x = 0$ and $x = 6$.)

   Solution. To obtain the area, integrate the difference in heights of the curves:

$$\text{Area} = \int_0^6 (2x - (x^2 - 4x))\,dx = \int_0^6 (6x - x^2)\,dx = \left[3x^2 - \frac{x^3}{3}\right]_0^6$$
$$= \left[x^2(3 - \frac{x}{3})\right]_0^6 = 36(3-2) - 0 = 36.$$

2. Show that
$$\int_1^4 \frac{1}{x + \sqrt{x}}\,dx = 2\ln(3/2).$$
   Hint: substitute $u = \sqrt{x}$.

   Solution. If $u = \sqrt{x}$, then $u^2 = x$, so $2u\,du = dx$. Also $u = 1$ when $x = 1$, and $u = 2$ when $x = 4$. Substituting into the integral gives that

$$\int_1^4 \frac{1}{x+\sqrt{x}}\,dx = \int_1^2 \frac{2u}{u^2+u}\,du = 2\int_1^2 \frac{1}{u+1}\,du = 2\ln(u+1)\Big|_1^2$$
$$= 2(\ln(3) - \ln(2)) = 2\ln(3/2).$$

3. The area bounded by the curves $y = \sin x$, $y = 0$, $x = 2\pi$, and $x = 3\pi$ is rotated about the y-axis. Show that the volume of the resulting solid equals $10\pi^2$.

   Hint: use the shell method, and integrate by parts.

   Solution. According to the shell method, one should integrate 2π times the distance to the rotation axis (in this case the distance equals x) times the height (in this case $\sin x$). Thus

$$\text{Volume} = 2\pi \int_{2\pi}^{3\pi} x \sin x\,dx.$$

To integrate by parts, set $u = x$ and $dv = \sin x\, dx$. Then $du = dx$ and $v = -\cos x$, so

$$\begin{aligned}\text{Volume} &= 2\pi\left[-x\cos x - \int(-\cos x)\,dx\right]_{2\pi}^{3\pi} = 2\pi\left[-x\cos x + \sin x\right]_{2\pi}^{3\pi}\\ &= 2\pi\big[(-3\pi\cos 3\pi + \sin 3\pi) - (-2\pi\cos 2\pi + \sin 2\pi)\big]\\ &= 2\pi[3\pi + 2\pi] = 10\pi^2.\end{aligned}$$

4. An artificial pond, created by damming a stream, is 10 meters deep, 200 meters wide, and 1,025 meters long. Show that the energy stored in the pond is approximately 10^{11} joules. In other words, if sluice gates are opened at the bottom of the dam, about 10^{11} J of work will be done by gravity in draining the pond.

   (The density of water is about 1,000 kg/m^3, and the acceleration of gravity is about 9.8 m/sec^2.)

   Solution. Suppose the coordinate y equals 0 at the bottom of the dam and equals 10 at the top of the dam. A "slice" of water of thickness Δy has volume $200 \times 1{,}025 \times \Delta y$ and weight $200 \times 1{,}025 \times 1{,}000 \times 9.8 \times \Delta y$. That slice moves a distance y to get to the bottom of the dam. The work done by gravity on that slice equals the product of the weight and the distance, or $200 \times 1{,}025 \times 1{,}000 \times 9.8 \times y \times \Delta y$. The total work done by gravity to drain the pond is therefore

$$200 \times 1{,}025 \times 1{,}000 \times 9.8 \times \int_0^{10} y\,dy.$$

   Since $\int_0^{10} y\,dy = \frac{1}{2}y^2\Big|_0^{10} = \frac{1}{2} \times 100$, the total work equals $200 \times 1{,}025 \times 1{,}000 \times 9.8 \times \frac{1}{2} \times 100 = 1.0045 \times 10^{11}$.

5. Do **either** part (a) **or** part (b), whichever you prefer.

   (a) The rational function $\dfrac{x^2+8}{x(x^2+4)}$ has the partial fractions decomposition $\dfrac{A}{x}+\dfrac{Bx+C}{x^2+4}$. Show that $A=2$.

   Solution. Clearing the denominators in the equation

   $$\frac{x^2+8}{x(x^2+4)} = \frac{A}{x} + \frac{Bx+C}{x^2+4}$$

   gives the identity

   $$x^2+8 = A(x^2+4) + (Bx+C)x.$$

   Now setting x equal to 0 shows that $8=4A$, whence $A=2$.

   (b) Use a trigonometric substitution to show that

   $$\int_0^1 \frac{1}{(4-x^2)^{3/2}}\,dx = \frac{\sqrt{3}}{12}.$$

   Solution. One can read off the substitution by drawing a right triangle with angle θ, hypotenuse 2, side x opposite the angle θ, and side $\sqrt{4-x^2}$ adjacent to the angle. Then $x=2\sin\theta$, so $dx = 2\cos\theta\,d\theta$, and $\sqrt{4-x^2} = 2\cos\theta$. When $x=0$, the angle $\theta=0$, and when $x=1$, the angle $\theta=\pi/6$. The integral becomes

   $$\begin{aligned}\int_0^{\pi/6} \frac{2\cos\theta}{(2\cos\theta)^3}\,d\theta &= \frac{1}{4}\int_0^{\pi/6} \sec^2\theta\,d\theta = \frac{1}{4}\tan\theta\Big|_0^{\pi/6} \\ &= \frac{1}{4}\left(\tan\frac{\pi}{6} - \tan 0\right) = \frac{1}{4}\cdot\frac{1}{\sqrt{3}} = \frac{\sqrt{3}}{12}.\end{aligned}$$