1. (a) Give an example of two infinite sets $A$ and $B$ such that $A \subset B$ (that is, $A$ is a proper subset of $B$) and such that there exists a bijective function $f\colon A \to B$.

   (b) Give an example of two infinite sets $X$ and $Y$ such that $X \subset Y$ and such that there does *not* exist a bijective function $g\colon X \to Y$.

2. Find integers $k$ and $n$ such that $1121k + 220n = 1$.

3. For each of the following sets, say what its cardinality is. Possible answers are $\aleph_0$ (countably infinite), $\mathbf{c}$ (cardinality of the continuum), and "other". You may assume the axiom of choice.

   (a) The set of prime numbers.

   (b) The set of real numbers of the form $r + s\sqrt{2}$, where the numbers $r$ and $s$ are rational numbers.

   (c) The set of all functions whose domain is the set of integers and whose codomain is the doubleton set $\{0, 1\}$.

4. The base-two logarithm function $\log_2$ is defined by the property

$$\log_2(x) = y \text{ if and only if } x = 2^y.$$

   Prove that $\log_2(220)$ is an irrational number.

5. This problem asks for two different proofs that the inequality $n < 2^n$ is true for every positive integer $n$.

   (a) Use the method of induction to prove the inequality.

   (b) Apply Cantor's theorem about power sets to prove the inequality.

## Extra credit

Prove Wilson's theorem about prime numbers and factorials: If $p$ is an integer greater than 1, then $p$ is a prime number if and only if

$$(p-1)! \equiv -1 \mod p.$$

### Remark

A so-called *Wilson prime* is a prime number $p$ that satisfies the stronger inequality $(p-1)! \equiv -1 \mod p^2$. Although only three examples of Wilson primes are known (5, 13, and 563), there is a conjecture that infinitely many Wilson primes exist.

Wilson's theorem is named after the eighteenth-century English mathematician John Wilson, but already around the year 1000 the theorem was known to the famous middle-Eastern scientist Abu Ali Hasan ibn al-Haitham, commonly known as Alhazen.