1. Let $A$ denote the set $\{2, \{\mathbb{Z}\}, \sqrt{3}\}$, and let $B$ denote the set $\{\varnothing, 5\}$. (As usual, $\mathbb{Z}$ denotes the set of integers, and $\varnothing$ denotes the empty set.) Find the following sets.
   (a) the intersection $A \cap B$
   (b) the union $A \cup B$
   (c) the power set of $B$
   (d) the Cartesian product $A \times B$

2. A binary operation $\oplus$ is defined on the set of positive real numbers $\mathbb{R}^+$ via
$$a \oplus b = \frac{1}{\frac{1}{a} + \frac{1}{b}} \qquad \text{for all } a \text{ and } b \text{ in } \mathbb{R}^+.$$
   (a) Is the operation $\oplus$ commutative?
   (b) Is the operation $\oplus$ associative?
   (c) Does the operation $\oplus$ have an identity element?

3. (a) Suppose that $f$ is a function whose domain is an interval $I$ and whose codomain is the set of real numbers. In calculus and real analysis, such a function $f$ is called *uniformly continuous* if
$$(\forall \epsilon > 0)\,(\exists\, \delta > 0)\,(\forall x \in I)\,(\forall y \in I)\,(|x-y| < \delta \implies |f(x)-f(y)| < \epsilon).$$
   Without using the symbol $\neg$ or the word "not", write a statement that is equivalent to the statement, "$f$ is not uniformly continuous".

   (b) In linear algebra, a set of vectors $\{v_1, \ldots, v_k\}$ is called *linearly dependent* if there exist constants $c_1, \ldots, c_k$, not all equal to zero, such that $c_1 v_1 + \cdots + c_k v_k = 0$. A set of vectors is called *linearly independent* if the set is not linearly dependent. Using the implication symbol $\implies$, write a statement that is equivalent to the statement, "the set of vectors $\{v_1, \ldots, v_k\}$ is linearly independent".

4. In each of the three parts, give an example of a set $A$, a set $B$, and a function $f\colon A \to B$ with the indicated property.
   (a) $f$ is injective but not surjective.
   (b) $f$ is surjective but not injective.
   (c) $f$ is invertible.

5. A consequence of the Euclidean algorithm for finding the greatest common divisor is an algorithm for finding the so-called *continued fraction* representation of a rational number. For example,

$$\frac{9}{43} = \cfrac{1}{4 + \cfrac{1}{1 + \cfrac{1}{3 + \cfrac{1}{2}}}} \qquad (*)$$

The representation $(*)$ results from the following computation:

$$\begin{aligned} 43 &= 4 \times 9 + 7 \\ 9 &= 1 \times 7 + 2 \\ 7 &= 3 \times 2 + 1 \end{aligned} \qquad \text{or} \qquad \begin{aligned} \frac{43}{9} &= 4 + \frac{7}{9} \\ \frac{9}{7} &= 1 + \frac{2}{7} \\ \frac{7}{2} &= 3 + \frac{1}{2} \end{aligned}$$

The left-hand column is the Euclidean algorithm, and repeated reciprocation in the equivalent right-hand column leads to $(*)$.

Generalizing from the above example, solve for the positive integers $x$ and $y$ in the following equation.

$$\frac{220}{2003} = \cfrac{1}{x + \cfrac{1}{9 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{y + \cfrac{1}{3}}}}}}$$

6. It is desired to prove that a certain open sentence $P(x)$ is true for every value of the real variable $x$. Student Orwell proposes the following method of proof:

   "Assuming the Axiom of Choice, there exists a well-ordering on the set of real numbers, say '$\ll$'. That is, every non-empty set of real numbers has a least element with respect to $\ll$. If we can show the following two steps

   (a) $P(1)$ is true;

   (b) for every $x$, the implication

   $$((\forall y < x)\, P(y)) \implies P(x)$$

   is true;

   then we can conclude that $P(x)$ is true for all $x$."

   Discuss Orwell's method: is it valid; is it incorrect but fixable; or is it fundamentally flawed?

7. On the first day of Christmas, my true love gave to me a partridge in a pear tree. On the second day of Christmas, my true love gave to me two turtle doves and a partridge in a pear tree. In general, on the $n$th day of Christmas, my true love gave to me $n + (n-1) + \cdots + 2 + 1$ items.

   (a) How many items did my true love give to me on the 12th day of Christmas?

   (b) How many items did my true love give to me in total (over the whole 12 days of Christmas)?

   (c) **Extra Credit**: Generalize your results by answering the analogues of parts (a) and (b) when the pattern continues for $N$ days rather than 12 days (where $N$ is an unspecified positive integer).