Recap from last time

A function f with domain A and codomain B is

- *surjective* if the image equals the codomain
  (you can also say that f is *onto*)
- *injective* if every two different elements of the domain have different images: $\forall a_1 \ \forall a_2$ if $a_1 \neq a_2$ then $f(a_1) \neq f(a_2)$.
  Logically equivalent is the contrapositive: if $f(a_1) = f(a_2)$ then $a_1 = a_2$.
  (you can also say that f is *one-to-one*)
- *bijective* if f is simultaneously injective and surjective
  (you can also say that f is a *one-to-one correspondence*)

Composite functions

What is the structure of $(\sin x)^2$?

If $f\colon A \to B$, and $g\colon B \to C$, then the composition of these two functions has domain A and codomain C and is given by the formula $g(f(a))$, abbreviated by $g \circ f$, or (in our textbook) simply gf.

Composing more functions gives more elaborate examples, like $\ln(1 + (\sin x)^2)$, written $khgf$ or $k \circ h \circ g \circ f$.

The algebra of function composition

Commutative law? Is $f \circ g$ the same as $g \circ f$? No, not in general.
$f\colon A \to B$, $g\colon B \to C$. One composition $g \circ f$ makes sense, but the other one $f \circ g$ does not make sense: the domains and codomains do not match up.
Even if both compositions make sense, they need not agree:
$(\sin x)^2$ is different from $\sin(x^2)$

Associative law? Is $(h \circ g) \circ f$ the same as $h \circ (g \circ f)$?
$f\colon A \to B$, $g\colon B \to C$, $h\colon C \to D$
Where does a typical element a go?
$(h \circ g) \circ f$ sends a to the same place that $h \circ g$ sends $f(a)$, namely, to $h(g(f(a)))$.
And the other function sends a to the same image.
So the associative law does hold for function composition.

The identity function $i \colon A \to A$ has the property that $i(a) = a$ for every a in the domain A.
The identity function acts as an identity element for function composition. That is, $f \circ i = f = i \circ f$.

If f and g are two functions for which $g \circ f$ makes sense and equals the identity function, then g and f are called *inverse functions*.
Example. $A = B =$ positive real numbers. $f(x) = x^2$. The inverse function is $g(x) = \sqrt{x}$.
Example. $A = B = \mathbf{R}$. $f(x) = x^2$. There is no inverse function, because $f(5) = f(-5)$, so there is not a unique way to undo the function.

To have an inverse function, a function must be bijective.