Vocabulary today

- *binary operation* on a set
- a set being *closed* under a binary operation

Binary operation

A binary operation on a set A is a special type of function: a function whose domain is the Cartesian product $A \times A$ and whose codomain is A.

Instead of writing $f(a, b)$, one often writes $a * b$.

Examples. $A = \mathbf{Z}$, $a * b = a + b$. Addition on the integers is a binary operation. Multiplication is another example. Division is not an example, because a/b is not necessarily an integer.
Example. $A = \mathbf{R} - \{0\}$. Division is a binary operation on this set.
Example. $A = \mathbf{Z}^+$ (positive integers), $a * b = a^b$.

Properties than a binary operation might or might not have

- commutative property: $a * b = b * a$
  (holds for addition and multiplication but not for exponentiation or division)
- associative property: $(a * b) * c = a * (b * c)$
  (holds for addition and multiplication but not for exponentiation or division)
- identity element: a special element e such that $a * e = a = e * a$ for every element a of the set A.
- If there is an identity element, then the question of inverse elements arises: does there exist an element a^{-1} such that $a * a^{-1} = e = a^{-1} * a$?
  If $*$ is addition, then a^{-1} is $-a$.

Closure

Example. The binary operation $+$ is defined on the integers. Restrict attention to the even integers. Does $+$ define a binary operation on the even integers? Yes.
Restrict attention to the odd integers. Does $+$ define a binary operation on the odd integers? No, because the sum of two odd integers is no longer odd.

If $*$ is a binary operation on a set A, and S is a subset of A, then S is called *closed* under the binary operation $*$ if for every s_1 and s_2 in S, the element $s_1 * s_2$ is still in S.