Announcement

Freshman/Sophomore Math Contest
Monday, April 10
Blocker 624
7:00–9:00 PM
Cash prizes!
http://www.math.tamu.edu/undergraduate/fscontest/

Order axioms

Suppose in addition to the algebraic axioms A1–A8 we have a distinguished subset P (for "positive") of nonzero elements that is closed under addition and under multiplication, and the trichotomy property holds: for each element x of the universe either $x = 0$ or $x \in P$ or $-x \in P$ (exclusive or).
These properties are A9–A10 in the textbook.

Example

- In the set of integers $\mathbb{Z}$, the positive set is $\mathbb{Z}^+$, the set $\{1, 2, 3, \ldots\}$.
- In the set of polynomials, we could take the positive set to be the polynomials whose highest power has a positive coefficient.

A positive subset corresponds to an order relation

Define "$a < b$" to mean "$b - a$ is an element of the positive set."
And "$b > a$" means the same thing.
Moreover "$a \leq b$" means "either $a = b$ or $a < b$."
And "$b \geq a$" means the same thing.

The relation $\leq$ is a linear ordering.

Example proof using the order relation

Number 5(a) on page 157:
$(a < 0) \wedge (b < 0) \implies ab > 0$.
Here is the consensus proof the groups settled on during class.

Proof.
Suppose $(a < 0) \wedge (b < 0)$.
By trichotomy, $-a > 0$ and $-b > 0$.
By closure, $(-a)(-b) > 0$.
By P5, $(-a)(-b) = ab$, so $ab > 0$, as required. □

(The proof in the back of the book confusingly cites Proposition 5.1.4, which does not exist. The authors meant to cite Q2 and Q3 of Proposition 5.1.5.)

Well ordering

A linearly ordered set A is called *well ordered* if every non-empty subset S has a least element: namely, an element ℓ of S such that $\ell \leq s$ for every element s of S (or $\ell < s$ for every element s of S that is different from ℓ).

Example

- $A = \mathbb{Z}^+$ (positive integers) is well ordered by $\leq$ (Axiom A11).
- $A = \mathbb{Z}$ (all integers) is not well ordered by $\leq$.
  The subset S consisting of negative integers has no least element.
- $A =$ the set of positive rational numbers. There is no least element because $\ell/2 < \ell$.