An example about prime numbers

Suppose $f(n) = n^2 + n + 41$. Let's make a table of some values.

n	1	2	3	4	5	6
$f(n)$	43	47	53	61	71	83

When n is a positive integer, is $f(n)$ always a prime number?
No, $f(41)$ is divisible by 41, hence is not prime.

How can we prove that a statement about positive integers is always true?

Example of the method of mathematical induction

How to prove that $n < 2^n$ for *every* positive integer n?

If there were a counterexample value of n, then by the well-ordering principle, there would be a smallest counterexample, say m.
Evidently $1 < 2^1$, so $m > 1$. Let k denote $m - 1$. Since k is a positive integer smaller than the least counterexample, $k < 2^k$.
Then $k + 1 < 2^k + 1 = 2^k + 2^0 < 2^k + 2^k = 2 \cdot 2^k = 2^{k+1}$. But $k + 1 = m$, so m is not a counterexample after all.
The contradiction shows that there cannot be a counterexample, so the inequality does hold for every positive integer.

Mathematical induction: the general strategy

To prove that a statement $P(n)$ holds for every positive integer n:

1. Prove that $P(1)$ is true (the *basis step*).
2. Prove that the *implication* $P(k) \implies P(k+1)$ holds for every positive integer k (the *induction step*).

Taken together, these two steps show that there cannot be a minimal criminal.
Therefore $P(n)$ must be true for every positive integer n.