# Examination 2

**Instructions:** Please write your solutions on your own paper. These problems should be treated as essay questions to answer in complete sentences.

Notation: The symbol $\mathbb{Z}$ denotes $\{ \dots, -1, 0, 1, 2, \dots \}$ (the set of all integers), and the symbol $\mathbb{N}$ denotes $\{ 1, 2, 3, \dots \}$ (the set of positive integers).

1. Fill in the blank with the appropriate word: A relation $R$ on a set $A$ is called a partial ordering if the relation $R$ is reflexive, ________________ , and antisymmetric.

2. Suppose $f : \mathbb{Z} \to \mathbb{N}$ is defined by the property that $f(n) = n^2 + 1$ for each integer $n$. Let the symbol $\mathbb{O}$ denote the set $\{ 1, 3, 5, \dots \}$ (the odd positive integers). Determine $f^{-1}(\mathbb{O})$ (that is, the inverse image of the set $\mathbb{O}$).

3. Let $R$ be the relation defined on $\mathbb{Z}$ by saying that $a$ is related to $b$ (in symbols, $aRb$) when the difference $a - b$ is an odd integer. Is this relation an equivalence relation? Explain why or why not.

4. Suppose $f : \mathbb{N} \to \mathbb{N}$ and $g : \mathbb{N} \to \mathbb{N}$ are defined by the properties that $f(n) = n^2$ and $g(n) = 2^n$ for each positive integer $n$. Let $h_1$ denote the composition $f \circ g$, and let $h_2$ denote the composition $g \circ f$. Which of the values $h_1(4)$ and $h_2(3)$ is the larger? Explain how you know.

5. Consider the binary operation $*$ on $\mathbb{Z}$ defined as follows: $m * n = m + n - mn$ when $m$ and $n$ are integers. Is the operation $*$ associative? Explain why or why not.

6. Consider the function $f : \mathbb{N} \to \mathbb{Z}$ defined as follows:

$$f(n) = \begin{cases} \dfrac{n}{2}, & \text{if } n \text{ is even,} \\ \dfrac{1-n}{2}, & \text{if } n \text{ is odd.} \end{cases}$$

   Show that $f$ is a bijection.

7. When $f : A \to A$ is a function whose domain and codomain are the same set $A$, there is an associated relation $R_f$ on $A$ defined by saying that $a$ is related to $b$ (in symbols, $aR_f b$) when $b = f(a)$. Show that symmetry of this relation $R_f$ means that the function $f$ is invertible and equal to its inverse function (in symbols, $f = f^{-1}$).