Examination 1

Instructions: Please write your solutions on your own paper. These problems should be treated as essay questions to answer in complete sentences.

1. Consider the following statement:

   Every polynomial of degree 3 has at least one real root.

   a) Identify the quantifiers (universal and existential) in the statement.

   b) Write the negation of the statement.

   Solution. The word "every" indicates a universal quantifier, and the words "at least one" represent an existential quantifier. The negation of the statement is, "Some polynomial of degree 3 has no real root."

2. Let $P(x)$ be the open sentence "$x \in A$," and let $Q(x)$ be the open sentence "$x \in B$." Suppose $A \subseteq B$ (that is, A is a subset of B). Complete the sentence

   $Q(x)$ is a ____________ condition for $P(x)$

   by filling in the blank with the appropriate word ("necessary" or "sufficient").

   Solution. $Q(x)$ is a necessary condition for $P(x)$. In other words, a necessary condition for x to be an element of the subset A is that x is an element of B. The condition is not sufficient if B has some elements that are outside A.

3. Rewrite the implication

   $$P \Rightarrow Q$$

   in a logically equivalent form using only the symbols $\neg$ and $\wedge$ (negation and conjunction), the letters P and Q, and parentheses. Explain your reasoning.

   Solution. The disjunction $(\neg P) \vee Q$ is logically equivalent to the implication. (Indeed, both statements are false precisely when P is true and Q is false.) Rewrite the disjunction with two canceling negations: namely,

   $$\neg\neg((\neg P) \vee Q).$$

   Distribute one of the negations, which flips the disjunction to a conjunction:

   $$\neg((\neg(\neg P)) \wedge \neg Q).$$

   But $\neg(\neg P)$ is equivalent to P, so the answer simplifies to $\neg(P \wedge \neg Q)$.

   You can check the answer by writing out a truth table.

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4. Consider the following true statement:

   If n is an even integer and m is an odd integer, then the product nm is an even integer.

   a) Write the contrapositive of the statement. Is it true? Explain why or why not.

   Solution. The contrapositive is, "If two integers n and m have an odd product nm, then either n is odd or m is even." The contrapositive is logically equivalent to the original true statement, so the contrapositive is true.

   You can also see directly that the contrapositive is true, for a product of two integers is odd if and only if both integers are odd. In that case, the disjunction "either n is odd or m is even" is true, since the clause "n is odd" is true.

   b) Write the converse of the statement. Is it true? Explain why or why not.

   Solution. The converse is, "If the product nm is even, then n is even and m is odd." The converse is a false statement. One counterexample is obtained by taking n to be 1 and m to be 2.

5. Suppose $A = \{ n \in \mathbb{Z} \mid n^3 \geq 5 \}$, and $B = \{ n \in \mathbb{Z} \mid n \geq 0 \}$. Which of the two sets $B - A$ (the complement of A in B) and $A - B$ (the complement of B in A) has more elements? Explain your reasoning.

   Solution. If n is negative, then so is n^3, so the set A contains no negative elements. Also $0^3 = 0 < 5$ and $1^3 = 1 < 5$, so the set A does not contain either 0 or 1. On the other hand, if $n \geq 2$, then $n^3 \geq 8 > 5$, so the set A contains all the integers that are greater than or equal to 2. Accordingly, the complement $B - A$ is the doubleton set $\{0, 1\}$, and the complement $A - B$ is the empty set. Thus $B - A$ has more elements than $A - B$.

6. State De Morgan's laws about complements of unions and intersections.

   Solution. See Theorem 2.2.4 on page 65 of the textbook.

7. Call a partition of a set *uniform* if all the subsets that are elements of the partition have the same cardinality as each other. For example, if $A = \{1, 2, 3\}$, then the set A admits precisely two uniform partitions: namely, the partition into three singleton sets, and the trivial partition whose only element is the set A itself.

   How many uniform partitions does the set $\{1, 2, 3, 4\}$ have? Explain your reasoning.

   Solution. There are five uniform partitions: one partition into four singletons, and three partitions into two doubletons, and the trivial partition whose only element is A itself.

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To see why there are three partitions into two doubletons, observe that the element 1 can be paired with either 2 or 3 or 4, and the remaining two elements form the complementary doubleton. Here is the complete list of uniform partitions.

$$\{\{1\},\{2\},\{3\},\{4\}\} \tag{1}$$

$$\{\{1,2\},\{3,4\}\} \tag{2}$$

$$\{\{1,3\},\{2,4\}\} \tag{3}$$

$$\{\{1,4\},\{2,3\}\} \tag{4}$$

$$\{A\} \tag{5}$$