Several Variable Calculus

Instructions These problems should be viewed as essay questions. Before making a calculation, you should explain in words what your strategy is.

Please write your solutions on your own paper. Each of the 10 problems counts for 10 points.

1. Let $\vec{v}$ be the vector whose $\hat{\imath}$, $\hat{\jmath}$, and $\hat{k}$ components are the last three digits of your student identification number. Find an equation for the plane perpendicular to $\vec{v}$ and passing through the point $(2, 2, 1)$.

   Solution. The numerical answer is different for each student. Here is a general solution assuming that the last three digits are a, b, and c. The vector $\langle a, b, c\rangle$ is normal to the plane, so the equation of the plane has the general form

   $$ax + by + cz = d.$$

   The value of d can be determined by plugging in the coordinates of the given point that lies in the plane:

   $$d = 2a + 2b + c.$$

   Thus one form of the answer is

   $$ax + by + cz = 2a + 2b + c.$$

   Another way to write the answer is

   $$a(x - 2) + b(y - 2) + c(z - 1) = 0.$$

2. Let $\vec{v}$ be the same vector as in the preceding problem. Find a vector that is orthogonal both to $\vec{v}$ and to the vector $\langle 5, 0, 1\rangle$.

   Solution. Of course the vector $\langle 0, 0, 0\rangle$ solves the problem! But presumably the goal is to find a "nontrivial" solution.

   One way to get a vector orthogonal to two specified vectors is to compute the cross product. If $\vec{v}$ is the vector $\langle a, b, c\rangle$, then the required vector is

   $$\langle a, b, c\rangle \times \langle 5, 0, 1\rangle.$$

This cross product can be worked out by the determinant formula:

$$\begin{vmatrix} \hat\imath & \hat\jmath & \hat k \\ a & b & c \\ 5 & 0 & 1 \end{vmatrix} = \hat\imath \begin{vmatrix} b & c \\ 0 & 1 \end{vmatrix} - \hat\jmath \begin{vmatrix} a & c \\ 5 & 1 \end{vmatrix} + \hat k \begin{vmatrix} a & b \\ 5 & 0 \end{vmatrix} = b\hat\imath - (a-5c)\hat\jmath - 5b\hat k.$$

The solution is not unique: any multiple of this vector is another solution.

3. The "twisted cubic" is the curve given by the parametric equations $x = t$, $y = t^2$, and $z = t^3$. Find parametric equations for the line tangent to this curve at the point where $t = 1$.

Solution. The coordinate vector of a point on the curve at time t is $\langle t, t^2, t^3\rangle$. The derivative vector $\langle 1, 2t, 3t^2\rangle$ points in the direction of the tangent line. When $t = 1$, this tangent vector becomes $\langle 1, 2, 3\rangle$. And when $t = 1$, the corresponding point on the curve has coordinates $(1, 1, 1)$.

Thus the problem amounts to writing equations for a line passing through the point $(1, 1, 1)$ in the direction of the vector $\langle 1, 2, 3\rangle$. Parametric equations for this line are

$$\begin{aligned} x &= 1 + t \\ y &= 1 + 2t \\ z &= 1 + 3t. \end{aligned}$$

The problem does not ask for the symmetric equations of the line; those equations would be

$$\frac{x-1}{1} = \frac{y-1}{2} = \frac{z-1}{3}.$$

4. Explain how to use a dot product to compute the distance between a plane and a line parallel to that plane.

Solution. Pick an arbitrary point on the plane and an arbitrary point on the line. Subtract the corresponding coordinates to get a vector pointing from one point to the other. Take the dot product of this vector with a unit vector in the direction of the normal to the plane. Finally, take the absolute value, if necessary, in case the normal vector is pointing backward.

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Another way to express this process is to find a vector joining a point on the plane to a point on the line and to compute the scalar projection of this vector onto the direction normal to the plane.

If the line and the plane are given by explicit equations, then all the data needed in this calculation can be read off from the equations. To get a point on the line (or on the plane), plug in arbitrary values (zero, for instance) for all the variables but one, and solve for the remaining variable. A vector normal to the plane with equation $ax+by+cz=d$ is $\langle a,b,c\rangle$. And you can make this vector into a unit vector by dividing it by its length $\sqrt{a^2+b^2+c^2}$.

5. The vector function $\vec{r}(t) = \langle 2\cos(t), 2\sin(t), t\rangle$ describes a helix (a spiral). Show that the curvature of this helix is constant (that is, the same for every value of t).

Solution. The details of the solution depend on which formula you apply for the curvature κ.

One formula says that if $\vec{T}$ is the unit tangent vector, namely $\vec{r}\,'/|\vec{r}\,'|$, then κ equals $|\vec{T}'|/|\vec{r}\,'|$. So compute:

$$\begin{aligned}
\vec{r}\,'(t) &= \langle -2\sin(t), 2\cos(t), 1\rangle,\\
|\vec{r}\,'(t)| &= \sqrt{4(\sin t)^2+4(\cos t)^2+1^2} = \sqrt{5},\\
\vec{T}(t) &= \frac{1}{\sqrt{5}}\left\langle -2\sin(t), 2\cos(t), 1\right\rangle,\\
\vec{T}'(t) &= \frac{1}{\sqrt{5}}\langle -2\cos(t), -2\sin(t), 0\rangle,\\
|\vec{T}'(t)| &= \frac{1}{\sqrt{5}}\sqrt{4(\cos t)^2+4(\sin t)^2} = \frac{2}{\sqrt{5}},\\
\kappa &= \frac{2/\sqrt{5}}{\sqrt{5}} = \frac{2}{5}.
\end{aligned}$$

Thus the curvature is indeed independent of t.

An alternative formula for κ is $\dfrac{|\vec{r}\,'\times\vec{r}\,''|}{|\vec{r}\,'|^3}$. Here is the computation using that formula:

$$\begin{aligned}
\vec{r}\,'(t) &= \langle -2\sin(t), 2\cos(t), 1\rangle,\\
\vec{r}\,''(t) &= \langle -2\cos(t), -2\sin(t), 0\rangle,\\
\vec{r}\,'(t)\times\vec{r}\,''(t) &= \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ -2\sin(t) & 2\cos(t) & 1 \\ -2\cos(t) & -2\sin(t) & 0 \end{vmatrix} = 2\sin(t)\hat{\imath} - 2\cos(t)\hat{\jmath} + 4\hat{k},\\
|\vec{r}\,'(t)\times\vec{r}\,''(t)| &= \sqrt{4(\sin t)^2 + 4(\cos t)^2 + 16} = \sqrt{20} = 2\sqrt{5},\\
|\vec{r}\,'(t)| &= \sqrt{4(\sin t)^2 + 4(\cos t)^2 + 1^2} = \sqrt{5},\\
\kappa &= \frac{2\sqrt{5}}{\left(\sqrt{5}\,\right)^3} = \frac{2}{5}.
\end{aligned}$$

Remark Even without doing any calculation, one can see by geometric reasoning that the curvature must be constant. Think of the helix as the threads on a bolt. Any point of the helix can be moved to any other point by rotating the bolt and moving it parallel to its axis. But a rigid motion does not change the curvature. Since all points of the helix are equivalent to each other via rigid motions, the curvature must be the same at all points.

6. Find a function $f(x, y)$ for which $\dfrac{\partial f}{\partial x} = 2x + y$ and $\dfrac{\partial f}{\partial y} = x + y$.

   Solution. This question is in the spirit of the game show *Jeopardy*: the answer to a partial-differentiation problem is given, and you need to recover the original function.

   The systematic method is to start by integrating the first equation with respect to x to deduce that

$$f(x, y) = x^2 + xy + C,$$

   where C is an integration constant. But from the point of view of the x partial derivative, what "constant" means is "independent of x," so the "constant" C is really a function $C(y)$ that depends on y. Next take the y partial derivative:

$$\frac{\partial f}{\partial y} = \frac{\partial}{\partial y}(x^2 + xy + C(y)) = x + C'(y).$$

On the other hand, it is given that $\dfrac{\partial f}{\partial y} = x + y$, so $C'(y)$ must be equal to y. Therefore $C(y) = \frac{1}{2}y^2 + k$, where this time k is authentically a constant (independent of both x and y). Accordingly, the final answer is that $f(x, y) = x^2 + xy + \frac{1}{2}y^2 + k$. Since the question asks for "a" function, not "the most general function," the answer $x^2 + xy + \frac{1}{2}y^2$ is perfectly acceptable.

Alternatively, you could solve the problem by the "guess and check" method. The simplest function whose x partial derivative equals $2x + y$ is $x^2 + xy$, and the simplest function whose y partial derivative equals $x+y$ is $xy+\frac{1}{2}y^2$. These two expressions do not match, but if you try merging them as the function $x^2 + xy + \frac{1}{2}y^2$, then you can easily check that this function solves the problem.

Yet another approach is to observe that the given first-order partial derivatives are first-degree expressions in x and y, so the original function $f(x, y)$ must be a second-degree expression in x and y. The most general such expression has the form $ax^2 + bxy + cy^2$ for certain numbers a, b, and c. Then

$$2x + y = \frac{\partial f}{\partial x} = 2ax + by,$$
$$x + y = \frac{\partial f}{\partial y} = bx + 2cy.$$

Evidently these equations for the undetermined coefficients a, b, and c are satisfied when $a = 1$, $b = 1$, and $c = 1/2$.

7. Suppose $z = \frac{1}{2}x^2 - \frac{1}{3}y^3$. Find the point(s) on this surface where the tangent plane is parallel to the plane with equation $x + y + z = 1$.

Solution. Two planes are parallel precisely when their normal vectors are parallel. The vector $\langle 1, 1, 1\rangle$ is normal to the given plane, so what needs to be determined is a vector normal to the surface. If the equation for the surface is rewritten in the form of a level surface as

$$z - \tfrac{1}{2}x^2 + \tfrac{1}{3}y^3 = 0,$$

then a normal vector can be obtained as the gradient vector $\langle -x, y^2, 1\rangle$. The two vectors $\langle 1, 1, 1\rangle$ and $\langle -x, y^2, 1\rangle$ have the same third coordinate, so these vectors are parallel precisely when the other two coordinates match up: namely, $1 = -x$ and $1 = y^2$. Consequently, $x = -1$ and $y = \pm 1$. Going back to the equation of the surface reveals that if $x = -1$ and $y = +1$, then $z = 1/6$; while if $x = -1$ and $y = -1$, then $z = 5/6$. Therefore the two points that solve the problem are $(-1, 1, 1/6)$ and $(-1, -1, 5/6)$.

8. Suppose $x\cos(y) + \sin(xz) = e^y$. Find $\dfrac{\partial z}{\partial x}$ at the point $(1, 0, \pi)$.

Solution. Differentiate implicitly with respect to x (viewing y as constant):

$$\cos(y) + \cos(xz)\frac{\partial}{\partial x}(xz) = 0, \quad \text{or} \quad \cos(y) + \cos(xz)\left(z + x\frac{\partial z}{\partial x}\right) = 0.$$

Insert the values of the coordinates at the specified point:

$$1 - \left(\pi + \frac{\partial z}{\partial x}\right) = 0, \qquad \text{so} \qquad \frac{\partial z}{\partial x} = 1 - \pi.$$

9. Suppose $f(x, y) = x^3 - 3xy + y^3$. Find the point(s) at which this function has a local minimum.

Solution. First identify the critical points by setting both partial derivatives equal to zero:

$$\begin{aligned} 0 &= \frac{\partial f}{\partial x} = 3x^2 - 3y, \\ 0 &= \frac{\partial f}{\partial y} = -3x + 3y^2. \end{aligned}$$

The first equation implies that $y = x^2$, and the second equation implies that $x = y^2 = (x^2)^2 = x^4$. Consequently, either $x = 0$ or $x = 1$; the corresponding values of y are 0 and 1. Thus the critical points are $(0, 0)$ and $(1, 1)$.

To apply the second-derivative test, compute the matrix of second-order partial derivatives:

$$\begin{pmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{pmatrix} = \begin{pmatrix} 6x & -3 \\ -3 & 6y \end{pmatrix}.$$

At the point $(0,0)$, this matrix becomes $\begin{pmatrix} 0 & -3 \\ -3 & 0 \end{pmatrix}$, which has negative determinant, so the point $(0,0)$ is a saddle point. At the point $(1,1)$, the matrix of second-order partial derivatives becomes $\begin{pmatrix} 6 & -3 \\ -3 & 6 \end{pmatrix}$. Now the determinant is positive, and both derivatives f_{xx} and f_{yy} are positive, so the point $(1,1)$ is a local minimum.

10. Suppose $f(x,y) = x^2 + y$. Find the absolute maximum value of this function on the closed disk where $x^2 + y^2 \leq 1$.

Solution. Observe that $\dfrac{\partial f}{\partial y} = 1$, so the function f has no critical points. Therefore all that is required is to find the maximum of f on the circle where $x^2 + y^2 = 1$.

Apply the method of Lagrange multipliers: the gradient of f, which equals $\langle 2x, 1\rangle$, must be parallel to the gradient of the function defining the circle, which is $\langle 2x, 2y\rangle$. In other words, there is a number λ such that

$$\begin{aligned} 2x &= 2x\lambda, \\ 1 &= 2y\lambda. \end{aligned}$$

The second equation says that $\lambda = 1/(2y)$, and substituting this expression into the first equation shows that $2x = 2x/(2y)$, or $2xy = x$, or $2xy - x = 0$, or $x(2y-1) = 0$. Therefore either $x = 0$ or $y = 1/2$.

If $x = 0$, the constraint that $x^2 + y^2 = 1$ implies that $y = \pm 1$. In this case, there are two candidate points, $(0,1)$ and $(0,-1)$. Now $f(0,1) = 1$ and $f(0,-1) = -1$. If $y = 1/2$, then the constraint that $x^2 + y^2 = 1$ implies that $x = \pm\sqrt{3}/2$. In this case, there are two candidate points, $(+\sqrt{3}/2, 1/2)$ and $(-\sqrt{3}/2, 1/2)$. The value of the function f at both of these points is $5/4$. The maximum value of the function f at the four candidate points is thus equal to $5/4$.

An alternative method is to observe first that $x^2 + y$ will not achieve a maximum when the number y is negative. Moreover, there is no need to consider negative values of x, since $f(-x,y) = f(x,y)$. Thus it suffices to find the maximum value of $f(x,y)$ on the part of the circle lying in the first quadrant. On that part of the circle, $y = \sqrt{1-x^2}$ (with the positive square root), so

$f(x, y)$ reduces to the single-variable function $x^2 + \sqrt{1 - x^2}$. The problem then becomes a question of maximizing this function of x when $0 \le x \le 1$.

Find the critical point of this one-variable function by setting the derivative equal to 0:

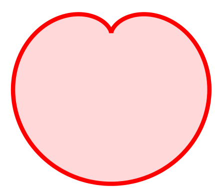
$$0 = 2x + \tfrac{1}{2}(1 - x^2)^{-1/2}(-2x).$$

Either $x = 0$ (which is an endpoint) or $0 = 2 - (1 - x^2)^{-1/2}$. The latter equation simplifies to $\sqrt{1 - x^2} = 1/2$ or $1 - x^2 = 1/4$ or $x^2 = 3/4$ or $x = \sqrt{3}/2$ (since x is assumed to be positive). The value of the function $x^2 + \sqrt{1 - x^2}$ at this critical point is $5/4$; the value at the endpoint where $x = 0$ is 1; and the value at the endpoint where $x = 1$ is 1. Thus the absolute maximum value is $5/4$.

Optional bonus problem for extra credit

The parametric equations

$$\begin{aligned} x &= (\sin t)(\sin \tfrac{1}{2}t)^2, \\ y &= (\cos t)(\sin \tfrac{1}{2}t)^2, \end{aligned} \qquad 0 \le t \le 2\pi$$



describe a *cardioid*, a heart-shaped figure.

Show that the arc length of this cardioid is equal to 4.

Solution. The arc length is the integral of the speed, which is

$$\sqrt{\left(\frac{\mathrm{d}x}{\mathrm{d}t}\right)^2 + \left(\frac{\mathrm{d}y}{\mathrm{d}t}\right)^2}.$$

First compute the derivatives using the product rule and the chain rule:

$$\begin{aligned} \frac{\mathrm{d}x}{\mathrm{d}t} &= (\cos t)(\sin \tfrac{1}{2}t)^2 + (\sin t)2(\sin \tfrac{1}{2}t)(\cos \tfrac{1}{2}t)(1/2), \\ \frac{\mathrm{d}y}{\mathrm{d}t} &= (-\sin t)(\sin \tfrac{1}{2}t)^2 + (\cos t)2(\sin \tfrac{1}{2}t)(\cos \tfrac{1}{2}t)(1/2). \end{aligned}$$

Of course the factors of 2 and $1/2$ cancel. Now add the squares of these derivatives, observing that the cross terms cancel out because of the minus sign in the second equation:

$$\begin{aligned}
\left(\frac{\mathrm{d}x}{\mathrm{d}t}\right)^2 + \left(\frac{\mathrm{d}y}{\mathrm{d}t}\right)^2 &= [(\cos t)^2 + (\sin t)^2](\sin \tfrac{1}{2}t)^4 \\
&\qquad + [(\sin t)^2 + (\cos t)^2](\sin \tfrac{1}{2}t)^2(\cos \tfrac{1}{2}t)^2 \\
&= (\sin \tfrac{1}{2}t)^4 + (\sin \tfrac{1}{2}t)^2(\cos \tfrac{1}{2}t)^2 \\
&= (\sin \tfrac{1}{2}t)^2[(\sin \tfrac{1}{2}t)^2 + (\cos \tfrac{1}{2}t)^2] \\
&= (\sin \tfrac{1}{2}t)^2.
\end{aligned}$$

Taking the square root shows that the speed is $\sin \frac{1}{2}t$. Therefore the computation of the arc length reduces to the following integral:

$$\int_0^{2\pi} \sin \tfrac{1}{2}t \, \mathrm{d}t = -2\cos \tfrac{1}{2}t \,\Big|_0^{2\pi} = -2(\cos \pi - \cos 0) = 4.$$