Several Variable Calculus

Instructions These problems should be viewed as essay questions. Before starting a calculation, you should explain your strategy in words.

Please write your solutions on your own paper. Each of the 10 problems counts for 10 points.

1. Find parametric equations for the line passing through the points $(2, 2, 1)$ and $(5, 0, 1)$.

   Solution. The vector joining the points is $\langle 5-2, 0-2, 1-1\rangle$, or $\langle 3, -2, 0\rangle$. Accordingly, parametric equations for the line can be written as follows:

   $$\begin{aligned} x &= 2 + 3t, \\ y &= 2 - 2t, \\ z &= 1, \end{aligned}$$

   where t is the parameter. Although the line is unique, the parametric equations are not unique. Another correct answer is

   $$\begin{aligned} x &= 5 + 3t, \\ y &= -2t, \\ z &= 1. \end{aligned}$$

2. Compute the volume of the parallelepiped determined by the three vectors $\langle 1, 0, 0\rangle$ and $\langle 2, 3, 0\rangle$ and $\langle 4, 5, 6\rangle$.

   Solution. The volume equals the magnitude of the scalar triple product of the three vectors, which can be computed by the following determinant:

   $$\begin{vmatrix} 1 & 0 & 0 \\ 2 & 3 & 0 \\ 4 & 5 & 6 \end{vmatrix} = \begin{vmatrix} 3 & 0 \\ 5 & 6 \end{vmatrix} + 0 + 0 = 18.$$

   Remark This calculation is the basis of the formula for changing variables in multiple integrals. Suppose new variables u, v, and w are related to x, y, and z as follows:

$$\begin{aligned} x &= u + 2v + 4w \\ y &= \phantom{u + {}} 3v + 5w \\ z &= \phantom{u + 2v + {}} 6w. \end{aligned}$$

Then the three given vectors in xyz-space correspond to the vectors $\langle 1, 0, 0\rangle$, $\langle 0, 1, 0\rangle$, and $\langle 0, 0, 1\rangle$ in uvw-space. The parallelepiped in xyz-space corresponds to a cube of side 1 in uvw-space, which has volume equal to 1. The Jacobian determinant

$$\frac{\partial(x, y, z)}{\partial(u, v, w)} = \begin{vmatrix} 1 & 2 & 4 \\ 0 & 3 & 5 \\ 0 & 0 & 6 \end{vmatrix} = 18$$

gives the magnification factor relating volume in uvw-space to volume in xyz-space.

3. Find an equation for the plane tangent to the paraboloid $z = x^2 + y^2$ at the point $(1, 2, 5)$.

Solution. Since $\left.\dfrac{\partial z}{\partial x}\right|_{(1,2,5)} = 2$ and $\left.\dfrac{\partial z}{\partial y}\right|_{(1,2,5)} = 4$, an equation for the tangent plane can be written as follows:

$$z - 5 = 2(x - 1) + 4(y - 2), \qquad \text{or} \qquad 5 = 2x + 4y - z.$$

An alternative approach is to rewrite the paraboloid in the form of a level surface $z - x^2 - y^2 = 0$ and compute a normal vector as the gradient:

$$\nabla \left(z - x^2 - y^2\right)\Big|_{(1,2,5)} = \langle -2x, -2y, 1\rangle\Big|_{(1,2,5)} = \langle -2, -4, 1\rangle.$$

Then the tangent plane can be written as follows:

$$-2(x - 1) - 4(y - 2) + 1(z - 5) = 0,$$

which again simplifies to $2x + 4y - z = 5$.

Several Variable Calculus

4. In which direction does the function $f(x, y, z) = y^2 - z^3$ increase most rapidly at the point $(2, 2, 1)$?

Solution. The function increases most rapidly in the direction of the gradient vector, and

$$\nabla f\Big|_{(2,2,1)} = \langle 0, 2y, -3z^2\rangle\Big|_{(2,2,1)} = \langle 0, 4, -3\rangle.$$

Usually a direction vector is taken to be a unit vector. Normalizing the vector above by dividing by the length gives the unit vector $\langle 0, \frac{4}{5}, \frac{-3}{5}\rangle$.

5. Determine the (absolute) maximum value of the function $f(x, y) = x + y^2$ on the closed disk where $x^2 + y^2 \le 4$.

Solution. Since $\nabla f = \langle 1, 2y\rangle$, and the first component is never equal to 0, there are no critical points of f inside the disk. Consequently, the problem reduces to finding the maximum value of $f(x, y)$ on the boundary circle, where $x^2 + y^2 = 4$.

According to the method of Lagrange multipliers, the maximum value on the boundary must occur at a point where ∇f is parallel to the gradient of the constraint function: namely, $\langle 2x, 2y\rangle$. The vectors $\langle 1, 2y\rangle$ and $\langle 2x, 2y\rangle$ are parallel when $x = 1/2$ (in which case the vectors are equal) and when $y = 0$.

On the boundary circle, if $y = 0$ then $x = \pm 2$, so $f(x, y) = \pm 2$. On the other hand, if $x = 1/2$ then $y^2 = 4 - (1/2)^2$, so $f(x, y) = (1/2) + 4 - (1/2)^2 = 17/4$. Of these candidate values of $f(x, y)$, the largest is $17/4$, so that value is the maximum of $f(x, y)$ on the closed disk.

An alternative method for the finding the maximum value on the boundary is to parametrize the circle by setting x equal to $2\cos(\theta)$ and y equal to $2\sin(\theta)$. Then $f(x, y)$ becomes $2\cos(\theta) + 4\sin^2(\theta)$. This single-variable function can be extremized by finding the critical points:

$$0 = \frac{d}{d\theta}\left(2\cos(\theta) + 4\sin^2(\theta)\right) = -2\sin(\theta) + 8\sin(\theta)\cos(\theta).$$

This equation implies that either $\sin(\theta) = 0$ or $\cos(\theta) = 1/4$. In the first case, $\cos(\theta) = \pm 1$, so the value of f becomes ± 2. In the second case, $\sin^2(\theta) = 1 - \cos^2(\theta) = 1 - (1/4)^2$, so the value of f becomes $(2/4) + 4(1 - (1/4)^2)$, or $17/4$. The conclusion again is that $17/4$ is the maximum value.

6. Compute the length of the curve $9y^2 = 4x^3$ starting at the point $(0, 0)$ and ending at the point $(1, \frac{2}{3})$.

Solution. One method is to rewrite the equation of the curve in explicit form as $y = (2/3)x^{3/2}$ (with the positive square root since the relevant part of the curve lies in the first quadrant). Then $\mathrm{d}y/\,\mathrm{d}x = x^{1/2}$, so $\sqrt{1 + (\mathrm{d}y/\,\mathrm{d}x)^2} = \sqrt{1+x}$, and the length of the curve equals

$$\int_0^1 \sqrt{1+x}\,\mathrm{d}x = \frac{2}{3}\left(1+x\right)^{3/2}\bigg|_0^1 = \frac{2}{3}\left(2\sqrt{2}-1\right).$$

Another method is to parametrize the curve via $x = t^2$ and $y = (2/3)t^3$, where $0 \le t \le 1$. Then $\mathrm{d}x/\,\mathrm{d}t = 2t$ and $\mathrm{d}y/\,\mathrm{d}t = 2t^2$, so

$$\sqrt{\left(\frac{\mathrm{d}x}{\mathrm{d}t}\right)^2 + \left(\frac{\mathrm{d}y}{\mathrm{d}t}\right)^2} = \sqrt{(2t)^2 + (2t^2)^2} = 2t\sqrt{1+t^2}.$$

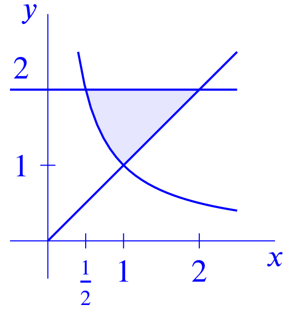
Therefore the length of the curve equals

$$\int_0^1 2t\sqrt{1+t^2}\,\mathrm{d}t = \frac{2}{3}\left(1+t^2\right)^{3/2}\bigg|_0^1 = \frac{2}{3}\left(2\sqrt{2}-1\right),$$

as before.

7. Evaluate the double integral $\iint_D y\,\mathrm{d}A$, where D denotes the region in the first quadrant lying above the hyperbola $xy = 1$ and the line $y = x$ and below the line $y = 2$.

Solution. This problem is Exercise 22 on page 863 in the Chapter 13 review. Here is a picture of the region:



Several Variable Calculus

The integral can be evaluated as an iterated integral:

$$\int_1^2 \int_{1/y}^y y\,\mathrm{d}x\,\mathrm{d}y = \int_1^2 y\left(y - \frac{1}{y}\right)\mathrm{d}y = \left[\frac{1}{3}y^3 - y\right]_1^2 = \frac{4}{3}.$$

It would be more complicated to integrate first with respect to y (for you would have to split the integral into two pieces).

8. Find the volume of the solid that lies above the paraboloid $z = x^2 + y^2$ and below the cone $z = \sqrt{x^2 + y^2}$.

Solution. The two surfaces intersect when $x^2 + y^2 = 1$, that is, when $z = 1$. Here is the set-up for the volume integral in cylindrical coordinates:

$$\begin{aligned} \int_0^{2\pi} \int_0^1 \int_{r^2}^r r\,\mathrm{d}z\,\mathrm{d}r\,\mathrm{d}\theta &= \int_0^{2\pi} \int_0^1 r\left(r - r^2\right)\mathrm{d}r\,\mathrm{d}\theta \\ &= \int_0^{2\pi} \left[\frac{1}{3}r^3 - \frac{1}{4}r^4\right]_0^1 \mathrm{d}\theta \\ &= \frac{\pi}{6}. \end{aligned}$$

9. Evaluate the line integral $\int_C \vec{F} \cdot \mathrm{d}\vec{r}$, where $\vec{F}(x, y) = y\hat{\imath} + x\hat{\jmath}$ and the curve C is given by $\vec{r}(t) = t^2\hat{\imath} - t^3\hat{\jmath}$, $0 \le t \le 1$.

Solution. The endpoints of the curve are $(0, 0)$ (when $t = 0$) and $(1, -1)$ (when $t = 1$). Since the vector field $\vec{F}$ is the gradient of the function xy (by inspection), the line integral equals $xy\Big|_{(0,0)}^{(1,-1)}$, or -1. Another way of phrasing this calculation is that

$$\int_C \vec{F} \cdot \mathrm{d}\vec{r} = \int_C y\,\mathrm{d}x + x\,\mathrm{d}y = \int_C \mathrm{d}(xy) = xy\Big|_{(0,0)}^{(1,-1)} = -1.$$

An alternative method is to use the parametrization of the curve:

$$\int_C y\,\mathrm{d}x + x\,\mathrm{d}y = \int_0^1 \left(-t^3\right)\left(2t\,\mathrm{d}t\right) + \left(t^2\right)\left(-3t^2\,\mathrm{d}t\right).$$

The integral simplifies to $\int_0^1 \left(-5t^4\right)\mathrm{d}t$, which evaluates to -1.

Several Variable Calculus

10. Evaluate the flux integral $\iint_S \vec{F} \cdot d\vec{S}$, where $\vec{F}(x, y, z) = x\hat{\imath} - z\hat{k}$ and the (open) surface S, oriented upward, is the part of the sphere $x^2 + y^2 + z^2 = 2$ that lies above the plane $z = 1$.

Solution.

Method 1 Since the surface is open, the divergence theorem is not directly applicable. But the divergence theorem can be brought into play by adding and subtracting the integral over a disk of radius 1 (oriented downward) that closes the bottom of the surface. Namely,

$$\begin{aligned} \iint_S \vec{F} \cdot \mathrm{d}\vec{S} &= \iint_{S \text{ plus disk}} \vec{F} \cdot \mathrm{d}\vec{S} - \iint_{\text{disk}} \vec{F} \cdot \mathrm{d}\vec{S} \\ &= \iiint_{\text{solid}} \left(\nabla \cdot \vec{F}\right) \mathrm{d}V - \iint_{\text{disk}} \vec{F} \cdot \mathrm{d}\vec{S}. \end{aligned}$$

Now $\nabla \cdot \vec{F} = 1 - 1 = 0$, so the problem reduces to $-\iint_{\text{disk}} \vec{F} \cdot \mathrm{d}\vec{S}$, where the disk is oriented in the $-\hat{k}$ direction. Since $\vec{F} \cdot \left(-\hat{k}\right) = z$, and $z = 1$ on the disk, the integral becomes

$$-\iint_{\text{disk of radius 1}} \mathrm{d}A, \qquad \text{or} \qquad -\pi.$$

Method 2 By inspection, the vector field $\vec{F}$ equals the curl of $-xz\hat{\jmath}$. By Stokes's theorem,

$$\iint_S \vec{F} \cdot \mathrm{d}\vec{S} = \iint_S \nabla \times \left(-xz\hat{\jmath}\right) \cdot \mathrm{d}\vec{S} = \int_C \left(-xz\hat{\jmath}\right) \cdot \mathrm{d}\vec{r},$$

where C is the curve bounding the surface: namely, a circle of radius 1 at height 1 (oriented counterclockwise since the surface is oriented upward). Accordingly, $z = 1$ on the curve, and $\left(-xz\hat{\jmath}\right) \cdot \mathrm{d}\vec{r} = -x\,\mathrm{d}y$ on C. The integral now equals $\int_{\text{circle}} -x\,\mathrm{d}y$, which by Green's theorem equals $-\iint_{\text{disk}} \mathrm{d}A$. Thus the answer is the negative of the area of a disk of radius 1, or $-\pi$.

Method 3 The integral can be computed directly from the definition of a surface integral. Rewrite the equation of the sphere in the form $z = f(x, y)$, where $f(x, y) = \sqrt{2 - x^2 - y^2}$. Then

$$\begin{aligned}\vec{F} \cdot \mathrm{d}\vec{S} &= \vec{F} \cdot \langle -f_x, -f_y, 1\rangle \,\mathrm{d}A \\ &= \left(\frac{x^2}{\sqrt{2 - x^2 - y^2}} - \sqrt{2 - x^2 - y^2} \right) \mathrm{d}A,\end{aligned}$$

where the integration with respect to dA takes place on a disk of radius 1. Convert to polar coordinates to get

$$\int_0^1 \int_0^{2\pi} \left(\frac{r^2 \cos^2(\theta)}{\sqrt{2 - r^2}} - \sqrt{2 - r^2} \right) r \,\mathrm{d}\theta \,\mathrm{d}r.$$

Computing the θ integral gives

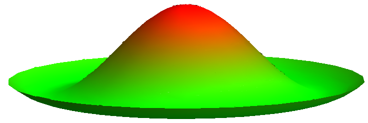
$$\pi \int_0^1 \left(\frac{r^2}{\sqrt{2 - r^2}} - 2\sqrt{2 - r^2} \right) r \,\mathrm{d}r.$$

Now change variables by setting u equal to $2 - r^2$ to get

$$\begin{aligned}\frac{\pi}{2} \int_1^2 \left(\frac{2 - u}{\sqrt{u}} - 2\sqrt{u} \right) \mathrm{d}u &= \frac{\pi}{2} \int_1^2 \left(\frac{2}{\sqrt{u}} - 3\sqrt{u} \right) \mathrm{d}u \\ &= \frac{\pi}{2} \left[4\sqrt{u} - 2u^{3/2} \right]_1^2 = -\pi.\end{aligned}$$

Optional bonus problem for extra credit

A sombrero is modeled by the equation $z = \cos(r)$ in cylindrical coordinates, where $0 \le r \le 4$.



If the density function is $\dfrac{1}{\sqrt{1 + \sin^2(r)}}$, find the center of mass of the sombrero.

Several Variable Calculus

Solution. Both the surface itself and the density function are independent of the polar angle θ, so the x and y coordinates of the center of mass must be equal to 0. Only the z coordinate needs to be computed.

The surface area element $\mathrm{d}S$ can be computed using your favorite coordinate system. Here is the computation using polar coordinates.

The coordinate vector is $\langle r\cos(\theta), r\sin(\theta), \cos(r)\rangle$, and what is needed is the length of the cross product vector

$$\begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ \cos(\theta) & \sin(\theta) & -\sin(r) \\ -r\sin(\theta) & r\cos(\theta) & 0 \end{vmatrix} = \hat{\imath} r\sin(r)\cos(\theta) + \hat{\jmath} r\sin(r)\sin(\theta) + \hat{k} r.$$

That length is $\sqrt{r^2\sin^2(r) + r^2}$, or $r\sqrt{1+\sin^2(r)}$. Thus the area element $\mathrm{d}S$ equals $r\sqrt{1+\sin^2(r)}\,\mathrm{d}r\,\mathrm{d}\theta$.

Accordingly, the mass of the sombrero equals

$$\int_0^{2\pi}\int_0^4 (\text{density})\, r\sqrt{1+\sin^2(r)}\,\mathrm{d}r\,\mathrm{d}\theta = \int_0^{2\pi}\int_0^4 r\,\mathrm{d}r\,\mathrm{d}\theta = 16\pi.$$

Since $z = \cos(r)$, the z coordinate of the center of mass equals

$$\frac{1}{16\pi}\int_0^{2\pi}\int_0^4 r\cos(r)\,\mathrm{d}r\,\mathrm{d}\theta = \frac{1}{8}\int_0^4 r\cos(r)\,\mathrm{d}r.$$

Integration by parts shows that the z coordinate of the center of mass equals

$$\frac{1}{8}\int_0^4 r\cos(r)\,\mathrm{d}r = \frac{1}{8}\Big[r\sin(r) + \cos(r)\Big]_0^4 = \frac{1}{8}\Big[4\sin(4) + \cos(4) - 1\Big]$$

(approximately -0.585).