Several Variable Calculus

1. Suppose $z = x\,e^y$. Find an equation of the plane tangent to this surface at the point $(1, 0, 1)$.

   Solution. This problem is Exercise 36 on page 790. One equation for the tangent plane is the following:

   $$z - 1 = \frac{\partial z}{\partial x}(1,0,1)(x-1) + \frac{\partial z}{\partial y}(1,0,1)(y-0).$$

   Now $\frac{\partial z}{\partial x} = e^y$, so $\frac{\partial z}{\partial x}(1,0,1) = 1$, while $\frac{\partial z}{\partial y} = x\,e^y$, so $\frac{\partial z}{\partial y}(1,0,1) = 1$. Substituting this information into the preceding equation yields the following result:

   $$z - 1 = (x-1) + y, \qquad \text{or} \qquad z = x + y.$$

   An alternative method is to rewrite the equation of the surface as follows: $z - x\,e^y = 0$. This equation displays the surface as a level surface, and the gradient vector $\langle -e^y, -x\,e^y, 1\rangle$ is normal to the surface. At the specified point, the gradient vector becomes $\langle -1, -1, 1\rangle$, so an equation for the tangent plane is the following:

   $$-x - y + z = d, \qquad \text{where} \qquad d = -1 - 0 + 1 = 0.$$

   This answer again says that $z = x + y$.

2. If $z = f(x, y)$ and $x = g(s, t)$ and $y = h(s, t)$, then z can be viewed as a function of s and t. Suppose at a certain point

   $$\frac{\partial f}{\partial x} = 2,\ \frac{\partial f}{\partial y} = 3,\ \frac{\partial g}{\partial s} = 5,\ \frac{\partial g}{\partial t} = 7,\ \frac{\partial h}{\partial s} = -1,\ \frac{\partial h}{\partial t} = -4.$$

   Use this information to determine the value of $\frac{\partial z}{\partial t}$.

   Solution. One method is to say that

   $$\frac{\partial z}{\partial t} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial t} = \frac{\partial f}{\partial x}\frac{\partial g}{\partial t} + \frac{\partial f}{\partial y}\frac{\partial h}{\partial t} = 2 \times 7 + 3 \times (-4) = 2.$$

   Alternatively, you could argue with differentials as follows:

   $$\mathrm{d}z = \frac{\partial f}{\partial x}\,\mathrm{d}x + \frac{\partial f}{\partial y}\,\mathrm{d}y = 2\,\mathrm{d}x + 3\,\mathrm{d}y.$$

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Similarly

$$\mathrm{d}x = \frac{\partial g}{\partial s}\,\mathrm{d}s + \frac{\partial g}{\partial t}\,\mathrm{d}t = 5\,\mathrm{d}s + 7\,\mathrm{d}t,$$
$$\mathrm{d}y = \frac{\partial h}{\partial s}\,\mathrm{d}s + \frac{\partial h}{\partial t}\,\mathrm{d}t = -\,\mathrm{d}s - 4\,\mathrm{d}t.$$

Substituting the expressions for $\mathrm{d}x$ and $\mathrm{d}y$ into the expression for $\mathrm{d}z$ shows that

$$\mathrm{d}z = (10\,\mathrm{d}s + 14\,\mathrm{d}t) + (-3\,\mathrm{d}s - 12\,\mathrm{d}t) = 7\,\mathrm{d}s + 2\,\mathrm{d}t.$$

Since $\mathrm{d}z = \frac{\partial z}{\partial s}\,\mathrm{d}s + \frac{\partial z}{\partial t}\,\mathrm{d}t$, it follows that $\frac{\partial z}{\partial s} = 7$ and $\frac{\partial z}{\partial t} = 2$.

3. Suppose $f(x, y, z) = z\,e^{xy}$. Find the direction in which $f(x, y, z)$ increases most rapidly at the point $(0, 1, 2)$.

Solution. This problem is Exercise 56 on page 791. The gradient vector points in the direction in which the function increases most rapidly. Now

$$\nabla f = \left\langle \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z} \right\rangle = \langle yz\,e^{xy}, xz\,e^{xy}, e^{xy} \rangle,$$

so $\nabla f(0, 1, 2) = \langle 2, 0, 1 \rangle$. The vector $\langle 2, 0, 1 \rangle$ is an acceptable answer, and so is the unit vector $\left\langle \frac{2}{\sqrt{5}}, 0, \frac{1}{\sqrt{5}} \right\rangle$.