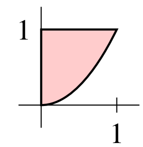
Several Variable Calculus

1. Find the y coordinate of the center of mass of a lamina in the first quadrant of the xy-plane bounded by the lines $x = 0$ and $y = 1$ and the parabola $y = x^2$. Suppose $\rho(x, y) = 1$ (the density function).



Solution. First compute the mass as an iterated double integral:

$$m = \int_0^1 \int_{x^2}^1 1\,\mathrm{d}y\,\mathrm{d}x = \int_0^1 (1 - x^2)\,\mathrm{d}x = \left[x - \frac{1}{3}x^3\right]_0^1 = \frac{2}{3}.$$

Next compute the y coordinate of the center of mass:

$$\bar{y} = \frac{1}{m}\int_0^1 \int_{x^2}^1 y\,\mathrm{d}y\,\mathrm{d}x = \frac{3}{2}\int_0^1 \left[\frac{1}{2}y^2\right]_{x^2}^1 \mathrm{d}x = \frac{3}{4}\int_0^1 (1 - x^4)\,\mathrm{d}x$$
$$= \frac{3}{4}\left[x - \frac{1}{5}x^5\right]_0^1 = \frac{3}{5}.$$

2. Find the surface area of the part of the plane $x + z = 2$ that lies above the square in the xy-plane with vertices $(0, 0)$, $(1, 0)$, $(1, 1)$, and $(0, 1)$.

Solution. The equation of the surface can be written in the form $z = 2 - x$. Then $\frac{\partial z}{\partial x} = -1$ and $\frac{\partial z}{\partial y} = 0$, so $\sqrt{1 + \left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2} = \sqrt{1 + 1 + 0} = \sqrt{2}$. Accordingly, the surface area equals

$$\int_0^1 \int_0^1 \sqrt{2}\,\mathrm{d}x\,\mathrm{d}y, \qquad \text{or} \qquad \sqrt{2}.$$

Remark Since the surface is a piece of a plane (not a curved surface), the area can actually be computed without using calculus. The surface is a rectangle sitting over a square and making an angle $\pi/4$ with the xy-plane. The area of this rectangle is the area of the square times the secant of the angle, hence 1 times $\sqrt{2}$.

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3. Evaluate the triple integral $\iiint_E yz\,\mathrm{d}V$, where E is the region defined by the inequalities $0 \le z \le 1$ and $0 \le y \le 2z$ and $0 \le x \le z+2$.

Solution. This problem is Exercise 7 on page 843 in section 13.8, one of the suggested exercises. There are different ways to set up the iterated integral. Here is one:

$$\begin{aligned}
\int_0^1 \int_0^{2z} \int_0^{z+2} yz\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z &= \int_0^1 \int_0^{2z} yz\,[x]_0^{z+2}\,\mathrm{d}y\,\mathrm{d}z \\
&= \int_0^1 \int_0^{2z} yz(z+2)\,\mathrm{d}y\,\mathrm{d}z \\
&= \int_0^1 z(z+2)\left[\frac{1}{2}y^2\right]_0^{2z}\mathrm{d}z \\
&= \int_0^1 z(z+2)\frac{4}{2}z^2\,\mathrm{d}z \\
&= 2\int_0^1 (z^4+2z^3)\,\mathrm{d}z \\
&= 2\left[\frac{1}{5}z^5+\frac{2}{4}z^4\right]_0^1 \\
&= \frac{7}{5}.
\end{aligned}$$