Math 304

Linear Algebra

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Solving a system of linear equations

An example

Solve the system of simultaneous equations

$$\begin{aligned} x_1 - x_2 + x_3 &= 9 \\ -2x_1 + 3x_2 + 10x_3 &= 3 \\ 3x_1 - 5x_2 - 16x_3 &= -5 \end{aligned}$$

Geometric interpretation: find the intersection points of three planes in three-dimensional space R^3.
In principle, there might be no solution (parallel planes) or exactly one solution (the planes intersect in one point) or infinitely many solutions (the planes intersect in a line).

Solution strategy: replace the system by a simpler *equivalent* system with the same solution(s).

Solving a system of linear equations

An example

$$\begin{aligned} x_1 - x_2 + x_3 &= 9 \\ -2x_1 + 3x_2 + 10x_3 &= 3 \\ 3x_1 - 5x_2 - 16x_3 &= -5 \end{aligned}$$

Add 2 times the first equation to the second equation to get the equivalent system

$$\begin{aligned} x_1 - x_2 + x_3 &= 9 \\ x_2 + 12x_3 &= 21 \\ 3x_1 - 5x_2 - 16x_3 &= -5 \end{aligned}$$

Solving a system of linear equations

An example

$$\begin{aligned} x_1 - x_2 + x_3 &= 9 \\ x_2 + 12x_3 &= 21 \\ 3x_1 - 5x_2 - 16x_3 &= -5 \end{aligned}$$

Add -3 times the first equation to the third equation to get the equivalent system

$$\begin{aligned} x_1 - x_2 + x_3 &= 9 \\ x_2 + 12x_3 &= 21 \\ -2x_2 - 19x_3 &= -32 \end{aligned}$$

Solving a system of linear equations

An example

$$\begin{aligned} x_1 - x_2 + x_3 &= 9 \\ x_2 + 12x_3 &= 21 \\ -2x_2 - 19x_3 &= -32 \end{aligned}$$

Add 2 times the second equation to the third equation to get the equivalent system

$$\begin{aligned} x_1 - x_2 + x_3 &= 9 \\ x_2 + 12x_3 &= 21 \\ 5x_3 &= 10 \end{aligned}$$

Solving a system of linear equations

An example

$$\begin{aligned} x_1 - x_2 + x_3 &= 9 \\ x_2 + 12x_3 &= 21 \\ 5x_3 &= 10 \end{aligned}$$

Divide the third equation by 5 to get the equivalent system

$$\begin{aligned} x_1 - x_2 + x_3 &= 9 \\ x_2 + 12x_3 &= 21 \\ x_3 &= 2 \end{aligned}$$

Solving a system of linear equations

An example

$$\begin{aligned} x_1 - x_2 + x_3 &= 9 \\ x_2 + 12x_3 &= 21 \\ x_3 &= 2 \end{aligned}$$

Take the value $x_3 = 2$ from the third equation and *back substitute* in the second equation to get $x_2 = -3$.
Then back substitute in the first equation to get $x_1 = 4$.

Thus our system of equations has a unique solution $(x_1, x_2, x_3) = (4, -3, 2)$.

Solving a system of linear equations

An example

Final check: verify that the values $(x_1, x_2, x_3) = (4, -3, 2)$ do work in all three of the original equations

$$\begin{aligned} x_1 - x_2 + x_3 &= 9 \\ -2x_1 + 3x_2 + 10x_3 &= 3 \\ 3x_1 - 5x_2 - 16x_3 &= -5 \end{aligned}$$

It checks; we are done.

Solving a system of linear equations

Reinterpretation

In the preceding calculation, the variables x_1, x_2, x_3 acted essentially as placeholders. Instead of working with the system

$$\begin{aligned} x_1 - x_2 + x_3 &= 9 \\ -2x_1 + 3x_2 + 10x_3 &= 3 \\ 3x_1 - 5x_2 - 16x_3 &= -5 \end{aligned}$$

we could work with the coefficient *matrix*

$$\left(\begin{array}{rrr|r} 1 & -1 & 1 & 9 \\ -2 & 3 & 10 & 3 \\ 3 & -5 & -16 & -5 \end{array}\right)$$

Solving a system of linear equations

Reinterpretation

The allowed *elementary row operations* on matrices are:

- add a multiple of a row to another row
- multiply (or divide) a row by a nonzero number
- interchange two rows

The process of using these operations to reduce a matrix to *echelon* (stair-step) form is called *Gaussian elimination*.

An example of Gaussian elimination

Exercise 5(l) on page 26

Solve

$$\begin{aligned} x_1 - 3x_2 + x_3 &= 1 \\ 2x_1 + x_2 - x_3 &= 2 \\ x_1 + 4x_2 - 2x_3 &= 1 \\ 5x_1 - 8x_2 + 2x_3 &= 5 \end{aligned}$$

The corresponding matrix is

$$\left(\begin{array}{rrr|r} 1 & -3 & 1 & 1 \\ 2 & 1 & -1 & 2 \\ 1 & 4 & -2 & 1 \\ 5 & -8 & 2 & 5 \end{array}\right)$$

An example of Gaussian elimination

Exercise 5(l) on page 26

$$\left(\begin{array}{rrr|r} 1 & -3 & 1 & 1 \\ 2 & 1 & -1 & 2 \\ 1 & 4 & -2 & 1 \\ 5 & -8 & 2 & 5 \end{array}\right) \xrightarrow{R_2 - 2R_1} \left(\begin{array}{rrr|r} 1 & -3 & 1 & 1 \\ 0 & 7 & -3 & 0 \\ 1 & 4 & -2 & 1 \\ 5 & -8 & 2 & 5 \end{array}\right)$$

$$\xrightarrow{R_3 - R_1} \left(\begin{array}{rrr|r} 1 & -3 & 1 & 1 \\ 0 & 7 & -3 & 0 \\ 0 & 7 & -3 & 0 \\ 5 & -8 & 2 & 5 \end{array}\right) \xrightarrow{R_4 - 5R_1} \left(\begin{array}{rrr|r} 1 & -3 & 1 & 1 \\ 0 & 7 & -3 & 0 \\ 0 & 7 & -3 & 0 \\ 0 & 7 & -3 & 0 \end{array}\right)$$

$$\xrightarrow[R_4 - R_2]{R_3 - R_2} \left(\begin{array}{rrr|r} 1 & -3 & 1 & 1 \\ 0 & 7 & -3 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right) \xrightarrow{R_2/7} \left(\begin{array}{rrr|r} 1 & -3 & 1 & 1 \\ 0 & 1 & -\frac{3}{7} & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

An example of Gaussian elimination

Exercise 5(l) on page 26

$$\left(\begin{array}{ccc|c} 1 & -3 & 1 & 1 \\ 0 & 1 & -\frac{3}{7} & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

Interpretation:
The variable x_3 is a *free variable* whose value can be prescribed arbitrarily; then the values of x_1 and x_2 are determined. The original system of equations has infinitely many solutions.

To write the solutions, it is convenient to do an extra step to bring the system to *reduced* row echelon form (Gauss-Jordan reduction).

An example of Gaussian elimination

Exercise 5(l) on page 26

$$\left(\begin{array}{ccc|c} 1 & -3 & 1 & 1 \\ 0 & 1 & -\frac{3}{7} & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right) \xrightarrow{R_1+3R_2} \left(\begin{array}{ccc|c} 1 & 0 & -\frac{2}{7} & 1 \\ 0 & 1 & -\frac{3}{7} & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

Now you can read off the solution

$$x_1 = 1 + \tfrac{2}{7}x_3$$
$$x_2 = \tfrac{3}{7}x_3$$
$$x_3 \quad \text{arbitrary}$$

The solution set represents a line in R^3.

An example of Gaussian elimination

Exercise 5(l) on page 26

Remark: In the computer program MATLAB, the following code produces the reduced row echelon form of the preceding example:

```
format rat;
A=[1 -3 1 1; 2 1 -1 2; 1 4 -2 1; 5 -8 2 5];
rref(A)
```

MATLAB's output is:

```
1            0            -2/7           1
0            1            -3/7           0
0            0             0             0
0            0             0             0
```