Math 304

Linear Algebra

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Recap of last time

Algorithms

- Gaussian elimination (echelon form)
- Gauss-Jordan reduction (reduced row echelon form)

Terminology

- equivalent linear systems
- elementary row operations
- back substitution
- augmented coefficient matrix

Matrix notation

Example. Suppose $A = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{pmatrix}$.

The notation a_{ij} denotes the matrix element in row i and column j. Thus $a_{23} = 6$ and a general formula is $a_{ij} = j + 3(i-1)$.

The matrix obtained by interchanging the rows and columns of A is the *transpose*, denoted A^T.

$$A^T = \begin{pmatrix} 1 & 4 \\ 2 & 5 \\ 3 & 6 \end{pmatrix}$$

The ij entry of A equals the ji entry of A^T.

A *square* matrix is called *symmetric* if it equals its transpose.

Linear combinations of matrices

Rectangular matrices of the same shape are added componentwise. Example:

$$\begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{pmatrix} + \begin{pmatrix} 0 & 1 & -1 \\ 2 & 7 & -9 \end{pmatrix} = \begin{pmatrix} 1 & 3 & 2 \\ 6 & 12 & -3 \end{pmatrix}$$

Multiplication of a matrix by a *scalar* is computed componentwise. Example:

$$2\begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{pmatrix} = \begin{pmatrix} 2 & 4 & 6 \\ 8 & 10 & 12 \end{pmatrix}$$

Multiplication of a matrix by a *matrix* follows a more complicated rule (coming up).

Interpreting a linear system as a vector equation

We can rewrite the example from yesterday

$$\begin{aligned} x_1 - x_2 + x_3 &= 9 \\ -2x_1 + 3x_2 + 10x_3 &= 3 \\ 3x_1 - 5x_2 - 16x_3 &= -5 \end{aligned}$$

as: $x_1 \begin{pmatrix} 1 \\ -2 \\ 3 \end{pmatrix} + x_2 \begin{pmatrix} -1 \\ 3 \\ -5 \end{pmatrix} + x_3 \begin{pmatrix} 1 \\ 10 \\ -16 \end{pmatrix} = \begin{pmatrix} 9 \\ 3 \\ -5 \end{pmatrix}.$

Solving the linear system amounts to writing the column vector $\begin{pmatrix} 9 \\ 3 \\ -5 \end{pmatrix}$ as a *linear combination* of the three column vectors $\begin{pmatrix} 1 \\ -2 \\ 3 \end{pmatrix}$, $\begin{pmatrix} -1 \\ 3 \\ -5 \end{pmatrix}$, and $\begin{pmatrix} 1 \\ 10 \\ -16 \end{pmatrix}$. [See Theorem 1.3.1, page 37.]

Product of a matrix and a vector

We *define* the product of a matrix and a vector in order to make

$$x_1 \begin{pmatrix} 1 \\ -2 \\ 3 \end{pmatrix} + x_2 \begin{pmatrix} -1 \\ 3 \\ -5 \end{pmatrix} + x_3 \begin{pmatrix} 1 \\ 10 \\ -16 \end{pmatrix} = \begin{pmatrix} 1 & -1 & 1 \\ -2 & 3 & 10 \\ 3 & -5 & -16 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}.$$

Example.

$$\begin{pmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{pmatrix} \begin{pmatrix} 7 \\ 8 \end{pmatrix} = \begin{pmatrix} 1 \times 7 + 2 \times 8 \\ 3 \times 7 + 4 \times 8 \\ 5 \times 7 + 6 \times 8 \end{pmatrix} = \begin{pmatrix} 23 \\ 53 \\ 83 \end{pmatrix}.$$

Product of matrices

We define the product of matrices to make the first matrix act independently on each column of the second matrix.

Example. $\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} \begin{pmatrix} 5 \\ 6 \end{pmatrix} = \begin{pmatrix} 17 \\ 39 \end{pmatrix}$ and $\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} \begin{pmatrix} 7 \\ 8 \end{pmatrix} = \begin{pmatrix} 23 \\ 53 \end{pmatrix}$,
so $\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} \begin{pmatrix} 5 & 7 \\ 6 & 8 \end{pmatrix} = \begin{pmatrix} 17 & 23 \\ 39 & 53 \end{pmatrix}.$

Warning! Matrix multiplication is *not commutative*.

$$\begin{pmatrix} 5 & 7 \\ 6 & 8 \end{pmatrix} \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} = \begin{pmatrix} 26 & 38 \\ 30 & 44 \end{pmatrix}$$

which is different from $\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} \begin{pmatrix} 5 & 7 \\ 6 & 8 \end{pmatrix}.$

Identity and inverse

The *additive* identity matrix has all entries equal to 0.

The *multiplicative* identity matrix I has entries equal to 0 except for the main diagonal, which has entries equal to 1. Example:

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \end{pmatrix} = a \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + b \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} + c \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} a \\ b \\ c \end{pmatrix}.$$

Two matrices are multiplicative *inverses* if their product (in either order) is the multiplicative identity matrix. Example:

$$\begin{pmatrix} 7 & 3 \\ 2 & 1 \end{pmatrix} \begin{pmatrix} 1 & -3 \\ -2 & 7 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & -3 \\ -2 & 7 \end{pmatrix} \begin{pmatrix} 7 & 3 \\ 2 & 1 \end{pmatrix}.$$

A matrix is *singular* if it does not have a multiplicative inverse.