Math 304
Linear Algebra

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Highlights

From last time:

- Solving a linear system is the same as writing a column vector as a linear combination of given column vectors.
- Matrix multiplication is not commutative.

Today:

- What are "elementary matrices"?
- Inverse matrices and solutions to linear systems.
- *LU* factorization of matrices.

Elementary matrices

Example. Multiplication by the matrix $\begin{pmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ does what?

$$\begin{pmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \end{pmatrix} = a \begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix} + b \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} + c \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} a \\ 2a+b \\ c \end{pmatrix},$$

so this matrix implements the elementary row operation of adding 2 times row 1 to row 2.

In general, applying an elementary row operation to the identity matrix produces an "elementary matrix" that implements that row operation.

Elementary matrices and inverse matrices

Example. Bring the matrix $A = \begin{pmatrix} 1 & 0 & 3 \\ 2 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ to reduced row echelon form by multiplying by elementary matrices.

$$\begin{pmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 3 \\ 2 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 3 \\ 0 & 1 & -6 \\ 0 & 0 & 1 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 6 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 3 \\ 2 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 3 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 0 & -3 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 6 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 3 \\ 2 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Elementary matrices and inverse matrices

Example continued

$$\begin{pmatrix}1&0&-3\\0&1&0\\0&0&1\end{pmatrix}\begin{pmatrix}1&0&0\\0&1&6\\0&0&1\end{pmatrix}\begin{pmatrix}1&0&0\\-2&1&0\\0&0&1\end{pmatrix}A=I$$

so A^{-1} equals the product of the elementary matrices

$$\begin{pmatrix}1&0&-3\\0&1&0\\0&0&1\end{pmatrix}\begin{pmatrix}1&0&0\\0&1&6\\0&0&1\end{pmatrix}\begin{pmatrix}1&0&0\\-2&1&0\\0&0&1\end{pmatrix},$$

namely $\begin{pmatrix}1&0&-3\\-2&1&6\\0&0&1\end{pmatrix}$.

In other words, the elementary row operations that turn the matrix A into the identity matrix also turn the identity matrix into the inverse matrix A^{-1}.

LU factorization

By tracking the elementary matrices that implement row operations, we can factor a matrix as a product of a lower triangular matrix and an upper triangular matrix.

Example. $A=\begin{pmatrix}1&2&3\\4&5&6\\7&8&9\end{pmatrix}$.

$$\begin{pmatrix}1&2&3\\4&5&6\\7&8&9\end{pmatrix}\xrightarrow[R_3-7R_1]{R_2-4R_1}\begin{pmatrix}1&2&3\\0&-3&-6\\0&-6&-12\end{pmatrix}\xrightarrow{R_3-2R_2}\begin{pmatrix}1&2&3\\0&-3&-6\\0&0&0\end{pmatrix}$$

$$\begin{pmatrix}1&0&0\\0&1&0\\0&-2&1\end{pmatrix}\begin{pmatrix}1&0&0\\0&1&0\\-7&0&1\end{pmatrix}\begin{pmatrix}1&0&0\\-4&1&0\\0&0&1\end{pmatrix}A=\begin{pmatrix}1&2&3\\0&-3&-6\\0&0&0\end{pmatrix}$$

$$A=\begin{pmatrix}1&0&0\\-4&1&0\\0&0&1\end{pmatrix}^{-1}\begin{pmatrix}1&0&0\\0&1&0\\-7&0&1\end{pmatrix}^{-1}\begin{pmatrix}1&0&0\\0&1&0\\0&-2&1\end{pmatrix}^{-1}\begin{pmatrix}1&2&3\\0&-3&-6\\0&0&0\end{pmatrix}$$

LU factorization

Example continued

$$A=\begin{pmatrix}1&0&0\\-4&1&0\\0&0&1\end{pmatrix}^{-1}\begin{pmatrix}1&0&0\\0&1&0\\-7&0&1\end{pmatrix}^{-1}\begin{pmatrix}1&0&0\\0&1&0\\0&-2&1\end{pmatrix}^{-1}\begin{pmatrix}1&2&3\\0&-3&-6\\0&0&0\end{pmatrix}$$

$$A=\begin{pmatrix}1&0&0\\4&1&0\\0&0&1\end{pmatrix}\begin{pmatrix}1&0&0\\0&1&0\\7&0&1\end{pmatrix}\begin{pmatrix}1&0&0\\0&1&0\\0&2&1\end{pmatrix}\begin{pmatrix}1&2&3\\0&-3&-6\\0&0&0\end{pmatrix}$$

$$A=\begin{pmatrix}1&0&0\\4&1&0\\7&2&1\end{pmatrix}\begin{pmatrix}1&2&3\\0&-3&-6\\0&0&0\end{pmatrix}:=LU.$$

We have factored the matrix A as the product of a lower triangular matrix L and an upper triangular matrix U.

Inverse matrices and solutions of linear systems

Suppose A is a square matrix, $\mathbf{x}$ is a vector of unknowns, and $\mathbf{b}$ is a column vector of numbers. The matrix equation $A\mathbf{x}=\mathbf{b}$ is equivalent to a linear system of simultaneous equations.

- If the matrix A is invertible, then the system has a unique solution: namely, $\mathbf{x}=A^{-1}\mathbf{b}$.
- If the matrix A is singular, then either the system is inconsistent or the system has infinitely many solutions.
- A *homogeneous* system (that is, $\mathbf{b}=\mathbf{0}$) is always consistent, so if A is invertible, a homogeneous system has the unique solution $\mathbf{x}=\mathbf{0}$, and if A is singular, a homogeneous system has infinitely many solutions.