Math 304

Linear Algebra

Harold P. Boas

boas@tamu.edu

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Highlights

From last time:

- Using row operations to find inverse matrices.
- Using elementary matrices to find the LU factorization of a matrix.
- Inverse matrices and solutions of linear systems.

Today:

- Determinants

Why determinants?

Associated to a square matrix A is a number, written $\det A$ or $|A|$, that detects whether the matrix is invertible.

- If $\det A \neq 0$, then the matrix A is invertible.
- If $\det A = 0$, then the matrix A is singular.

The case of 2×2 matrices.

$$\det \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc.$$

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}^{-1} = \frac{1}{ad - bc} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix} \qquad \text{if } ad - bc \neq 0.$$

Properties of determinants of $n \times n$ matrices

- The determinant of the identity matrix equals 1.
- More generally, the determinant of a triangular matrix (either upper triangular or lower triangular) equals the product of the numbers on the diagonal.
- Interchanging two rows of a matrix changes the sign of the determinant.
- The determinant is multiplicative: $\det(AB) = \det(A)\det(B)$.

The first three properties tell us the determinant of any elementary matrix.
The fourth property then tells us how the determinant changes under elementary row operations.

Computing a determinant using row operations

Example: exercise 2a on page 103

$$\begin{vmatrix} 0 & 1 & 2 & 3 \\ 1 & 1 & 1 & 1 \\ -2 & -2 & 3 & 3 \\ 1 & 2 & -2 & -3 \end{vmatrix} \overset{R_1 \leftrightarrow R_2}{=} - \begin{vmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & 3 \\ -2 & -2 & 3 & 3 \\ 1 & 2 & -2 & -3 \end{vmatrix}$$

$$\overset{R_3+2R_1}{=} - \begin{vmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 5 & 5 \\ 1 & 2 & -2 & -3 \end{vmatrix} \overset{R_4-R_1}{=} - \begin{vmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 5 & 5 \\ 0 & 1 & -3 & -4 \end{vmatrix}$$

$$\overset{R_4-R_2}{=} - \begin{vmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 5 & 5 \\ 0 & 0 & -5 & -7 \end{vmatrix} \overset{R_4+R_3}{=} - \begin{vmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 5 & 5 \\ 0 & 0 & 0 & -2 \end{vmatrix}$$

$$\overset{\text{triangular}}{=} -1 \times 1 \times 5 \times (-2) = 10.$$

Cofactor expansion

If a_{ij} is an element of a square matrix A, the corresponding *minor* is the determinant of the matrix that remains when row i and column j are deleted.

Example. If $A = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix}$, then the minor of the element 1 is $\begin{vmatrix} 5 & 6 \\ 8 & 9 \end{vmatrix} = -3$, the minor of 2 is $\begin{vmatrix} 4 & 6 \\ 7 & 9 \end{vmatrix} = -6$, and the minor of 3 is $\begin{vmatrix} 4 & 5 \\ 7 & 8 \end{vmatrix} = -3$.

The *cofactor* of a_{ij} is the minor multiplied by a plus sign if $i+j$ is even and a minus sign if $i+j$ is odd. In the example, the minors of 1 and 3 are both -3, and the minor of 2 is $-(-6) = 6$.

You can compute the determinant by picking a row, multiplying each element in that row by its cofactor, and adding the results.

Thus $\det A = 1 \times (-3) + 2 \times 6 + 3 \times (-3) = 0$.

More on cofactor expansion

You can expand a determinant along any row, and since $\det(A^T) = \det A$, you can also expand a determinant along any column.

Example. (exercise 5 on page 97, using the first column)

$$\begin{vmatrix} a-x & b & c \\ 1 & -x & 0 \\ 0 & 1 & -x \end{vmatrix} = (a-x)\begin{vmatrix} -x & 0 \\ 1 & -x \end{vmatrix} - \begin{vmatrix} b & c \\ 1 & -x \end{vmatrix} + 0 \begin{vmatrix} b & c \\ -x & 0 \end{vmatrix}$$

$$= (a-x)x^2 - (-bx - c) = -x^3 + ax^2 + bx + c$$

We will see more examples like this when we study eigenvalues in Chapter 6.