Math 304

Linear Algebra

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Highlights

Reminder. The first examination is Friday, June 9.

From last time:

- A set of vectors is linearly independent if none of the vectors is in the span of the others.
- A set of vectors is linearly dependent if some non-trivial linear combination of the vectors equals the $\mathbf{0}$ vector.

Today:

- the notions of basis and dimension
- changing coordinates

Basis

A *basis* for a vector space is:

- a linearly independent set of vectors that spans the vector space
- equivalently, a maximal linearly independent set of vectors
- equivalently, a minimal spanning set for the vector space

Example. The *standard basis* for R^3 consists of the three vectors $\mathbf{e}_1 = \begin{pmatrix}1\\0\\0\end{pmatrix}$, $\mathbf{e}_2 = \begin{pmatrix}0\\1\\0\end{pmatrix}$, and $\mathbf{e}_3 = \begin{pmatrix}0\\0\\1\end{pmatrix}$.

One nonstandard basis for R^3 consists of the three vectors $\mathbf{v}_1 = \begin{pmatrix}1\\2\\3\end{pmatrix}$, $\mathbf{v}_2 = \begin{pmatrix}4\\5\\6\end{pmatrix}$, and $\mathbf{v}_3 = \begin{pmatrix}7\\8\\0\end{pmatrix}$.

Example: exercise 10 on page 151

Extract a basis for R^3 from the following vectors:

$$\mathbf{x}_1 = \begin{pmatrix}1\\2\\2\end{pmatrix}, \quad \mathbf{x}_2 = \begin{pmatrix}2\\5\\4\end{pmatrix}, \quad \mathbf{x}_3 = \begin{pmatrix}1\\3\\2\end{pmatrix}, \quad \mathbf{x}_4 = \begin{pmatrix}2\\7\\4\end{pmatrix}, \quad \mathbf{x}_5 = \begin{pmatrix}1\\1\\0\end{pmatrix}$$

Solution. The vectors $\mathbf{x}_1$ and $\mathbf{x}_2$ are linearly independent, but they do not form a basis for R^3 because they do not span R^3. The vectors $\mathbf{x}_1$, $\mathbf{x}_2$, and $\mathbf{x}_3$ do not form a basis because they are linearly dependent ($\mathbf{x}_3 = \mathbf{x}_2 - \mathbf{x}_1$). Vectors $\mathbf{x}_1$, $\mathbf{x}_2$, and $\mathbf{x}_4$ do not form a basis because they are linearly dependent: $\det \begin{pmatrix}1&2&2\\2&5&7\\2&4&4\end{pmatrix} = 0$. However, $\begin{pmatrix}1&2&1\\2&5&1\\2&4&0\end{pmatrix} = -2 \neq 0$, so vectors $\mathbf{x}_1$, $\mathbf{x}_2$, and $\mathbf{x}_5$ do form a basis for R^3.

Dimension

The *dimension* of a vector space is the number of vectors in a basis. (All bases of a vector space have the same number of vectors.)

Examples.

- The dimension of R^3 equals 3.
- The dimension of the space $R^{2\times 5}$ of 2×5 matrices equals 10.
- The dimension of the space P_3 of polynomials of degree less than 3 equals 3. (One basis is $1, x, x^2$.)
- The function space $C[0,1]$ is infinite dimensional. (Any finite subset of the functions $1, x, x^2, x^3, \ldots$ is linearly independent.)

Change of basis: an example

Let $\mathbf{u}_1 = \begin{pmatrix}1\\1\end{pmatrix}$ and $\mathbf{u}_2 = \begin{pmatrix}-1\\1\end{pmatrix}$ be a nonstandard basis for R^2,

and let $\mathbf{e}_1$ and $\mathbf{e}_2$ be the standard basis vectors. Let $\mathbf{x} = \begin{pmatrix}5\\3\end{pmatrix}$.

Then $\mathbf{x} = 5\mathbf{e}_1 + 3\mathbf{e}_2$. In other words, 5 and 3 are the *coordinates* of the vector $\mathbf{x}$ with respect to the standard basis. Since $\mathbf{x} = 4\mathbf{u}_1 - \mathbf{u}_2$, the coordinates of $\mathbf{x}$ with respect to the $\mathbf{u}$ basis are 4 and -1.

Question. How are the $\mathbf{u}$ coordinates and the standard coordinates related?

$$\begin{pmatrix}5\\3\end{pmatrix}_{\mathbf{e}} = \mathbf{x} = 4\mathbf{u}_1 - \mathbf{u}_2 = \begin{pmatrix}1 & -1\\1 & 1\end{pmatrix}\begin{pmatrix}4\\-1\end{pmatrix}_{\mathbf{u}}$$

The *transition matrix* between $\mathbf{u}$ coordinates and standard coordinates is the matrix whose columns are the standard coordinates of the $\mathbf{u}$ basis vectors. The inverse matrix gives the change from standard coordinates to $\mathbf{u}$ coordinates.

Example continued

Consider another nonstandard basis $\mathbf{v}_1 = \begin{pmatrix}5\\6\end{pmatrix}$, $\mathbf{v}_2 = \begin{pmatrix}4\\5\end{pmatrix}$.

How are coordinates with respect to the $\mathbf{u}$ basis related to coordinates with respect to the $\mathbf{v}$ basis?

The matrix $\begin{pmatrix}5 & 4\\6 & 5\end{pmatrix}$ whose columns are $\mathbf{v}_1$ and $\mathbf{v}_2$ transforms $\mathbf{v}$ coordinates to standard coordinates, and the inverse matrix $\begin{pmatrix}5 & -4\\-6 & 5\end{pmatrix}$ transforms from standard coordinates to $\mathbf{v}$ coordinates. Since $\begin{pmatrix}1 & -1\\1 & 1\end{pmatrix}$ transforms from $\mathbf{u}$ coordinates to standard coordinates, the product matrix

$$\begin{pmatrix}5 & -4\\-6 & 5\end{pmatrix}\begin{pmatrix}1 & -1\\1 & 1\end{pmatrix} = \begin{pmatrix}1 & -9\\-1 & 11\end{pmatrix}$$

transforms from $\mathbf{u}$ coordinates to $\mathbf{v}$ coordinates.

So the $\mathbf{v}$ coordinates of $\mathbf{x}$ are $\begin{pmatrix}1 & -9\\-1 & 11\end{pmatrix}\begin{pmatrix}4\\-1\end{pmatrix}_{\mathbf{u}} = \begin{pmatrix}13\\-15\end{pmatrix}_{\mathbf{v}}$.