Math 304

Linear Algebra

Harold P. Boas

boas@tamu.edu

June 16, 2006

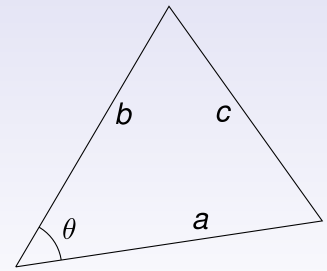
Highlights

From last time:

- matrix representations of linear transformations
- similar matrices

Today:

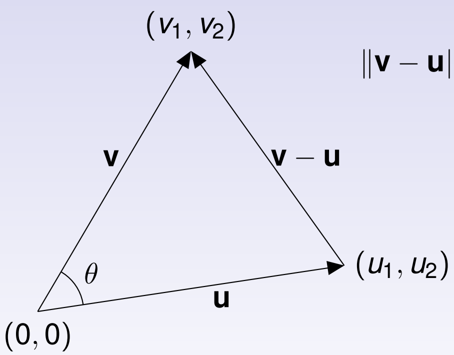
- applications of the law of cosines



$$c^2 = a^2 + b^2 - 2ab\cos(\theta)$$

Law of cosines for vectors

If $\mathbf{u} = \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}$ is a vector in R^2, then the *length* of $\mathbf{u}$, written $\|\mathbf{u}\|$, equals $\sqrt{u_1^2 + u_2^2}$ (Pythagorean theorem). Then:



$$\begin{aligned} \|\mathbf{v} - \mathbf{u}\|^2 &= \|\mathbf{u}\|^2 + \|\mathbf{v}\|^2 - 2\|\mathbf{u}\|\,\|\mathbf{v}\|\cos(\theta) \\ &= (v_1 - u_1)^2 + (v_2 - u_2)^2 \\ &= \|\mathbf{u}\|^2 + \|\mathbf{v}\|^2 - 2(u_1 v_1 + u_2 v_2) \end{aligned}$$

So $\|\mathbf{u}\|\,\|\mathbf{v}\|\cos(\theta) = u_1 v_1 + u_2 v_2 \stackrel{\text{def}}{=}$ *scalar product* of $\mathbf{u}$ and $\mathbf{v}$.

Notation

Some notations for the scalar product $u_1 v_1 + u_2 v_2$ of vectors $\mathbf{u}$ and $\mathbf{v}$ are:

- $\mathbf{u} \cdot \mathbf{v}$ (scalar product = “dot product”)
- $\langle \mathbf{u}, \mathbf{v} \rangle$
- $\mathbf{u}^T \mathbf{v} = \begin{pmatrix} u_1 & u_2 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}$ (our book’s notation)

In R^3 or R^n, the notation is analogous. One still has the basic formula for the angle θ between two vectors $\mathbf{u}$ and $\mathbf{v}$:

$$\cos(\theta) = \frac{\mathbf{u}^T \mathbf{v}}{\|\mathbf{u}\|\,\|\mathbf{v}\|}$$

and the *Cauchy-Schwarz inequality*: $|\mathbf{u}^T \mathbf{v}| \leq \|\mathbf{u}\|\,\|\mathbf{v}\|$.

Orthogonality and planes

Example. For which value of the parameter a will the vectors $\mathbf{u} = \begin{pmatrix} a \\ 2 \\ 3 \end{pmatrix}$ and $\mathbf{v} = \begin{pmatrix} 4 \\ 5 \\ a \end{pmatrix}$ in R^3 be orthogonal (that is, perpendicular)?

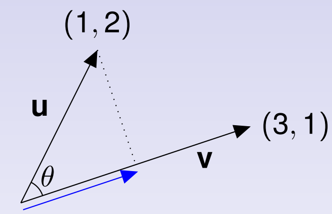
Solution. We want $\theta = 90°$, so $\cos(\theta) = 0$, so the scalar product $\mathbf{u}^T\mathbf{v} = 0$. Therefore $4a + 10 + 3a = 0$, so $a = -10/7$.

Example. Write an equation for the plane in R^3 passing through the origin with *normal* (perpendicular) vector $\mathbf{N} = (3, -1, 7)^T$.

Solution. A point (x, y, z) lies on the plane if $(x, y, z)^T \perp \mathbf{N}$. The orthogonality condition gives the equation $3x - y + 7z = 0$.

Projections

Example. Find the *projection* of the vector $\mathbf{u} = (1, 2)^T$ onto (the direction of) the vector $\mathbf{v} = (3, 1)^T$.



Solution. The *scalar* projection (signed length) equals $\|\mathbf{u}\| \cos(\theta)$, which is the same as the scalar product $\mathbf{u}^T \frac{\mathbf{v}}{\|\mathbf{v}\|}$. To get the *vector* projection, multiply this length by the *unit* vector $\frac{\mathbf{v}}{\|\mathbf{v}\|}$. Thus

$$(\text{projection of } \mathbf{u} \text{ onto } \mathbf{v}) = \left(\mathbf{u}^T \frac{\mathbf{v}}{\|\mathbf{v}\|}\right) \frac{\mathbf{v}}{\|\mathbf{v}\|} = \left(\frac{3}{2}, \frac{1}{2}\right)^T.$$

Projection and the distance to a plane

Example. Find the distance in R^3 from the point P with coordinates $(2, 1, 2)$ to the plane with equation $4x + 7y + 4z = 5$.

Solution. By inspection, you can see that one particular point on the plane is $(1, -1, 2)$. An equivalent equation for the plane is $4(x-1) + 7(y+1) + 4(z-2) = 0$. Thus $\mathbf{N} = (4, 7, 4)^T$ gives a vector normal to the plane.
The vector $\mathbf{v} = (2, 1, 2)^T - (1, -1, 2)^T = (1, 2, 0)^T$ joins a particular point in the plane to the point P, but not along a perpendicular. You can get the perpendicular distance by taking the length of the projection of $\mathbf{v}$ on the normal $\mathbf{N}$, namely $\left|\mathbf{v}^T \frac{\mathbf{N}}{\|\mathbf{N}\|}\right| = \frac{4+14+0}{\sqrt{16+49+16}} = 2.$