Math 304

Linear Algebra

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Highlights

From last time:

- To find the least-squares solution to $A\mathbf{x} = \mathbf{b}$, solve the related problem $A^T A\mathbf{x} = A^T\mathbf{b}$.

Today:

- inner products and norms

Inner product = generalization of scalar product

In R^2, the standard scalar product of vectors $\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$ and $\begin{pmatrix} y_1 \\ y_2 \end{pmatrix}$ is $x_1 y_1 + x_2 y_2$.
In an anisotropic problem, one might want to use a *weighted* scalar product, such as $2x_1 y_1 + 5x_2 y_2$. This will change the notions of length and angle.

In general, an *inner product* $\langle \mathbf{x}, \mathbf{y} \rangle$ must satisfy three properties:

- symmetry: $\langle \mathbf{x}, \mathbf{y} \rangle = \langle \mathbf{y}, \mathbf{x} \rangle$
- linearity: $\langle \mathbf{x} + \mathbf{y}, \mathbf{z} \rangle = \langle \mathbf{x}, \mathbf{z} \rangle + \langle \mathbf{y}, \mathbf{z} \rangle$ and $\langle c\mathbf{x}, \mathbf{y} \rangle = c\langle \mathbf{x}, \mathbf{y} \rangle$ for every scalar c
- positivity: $\langle \mathbf{x}, \mathbf{x} \rangle$ is positive unless $\mathbf{x} = \mathbf{0}$.

The *norm* associated to an inner product is given by $\|\mathbf{x}\| = \sqrt{\langle \mathbf{x}, \mathbf{x} \rangle}$.

Examples of inner products

- The standard scalar product on R^3 is the basic example of an inner product.
- On a vector space of functions, a common inner product is integration: $\langle f, g \rangle = \int_a^b f(x)g(x)\,dx$.

Example. Show that for the inner product corresponding to integration over the interval $[-1, 1]$, the Legendre polynomials 1, x, and $\frac{1}{2}(3x^2 - 1)$ are orthogonal to each other.

Solution. Show that the inner products are equal to 0.
$\langle 1, x \rangle = \int_{-1}^{1} x\,dx = 0$ by symmetry.
$\langle 1, \frac{1}{2}(3x^2 - 1) \rangle = \int_{-1}^{1} \frac{1}{2}(3x^2 - 1)\,dx = \frac{1}{2}\left[x^3 - x\right]_{-1}^{1} = 0.$
$\langle x, \frac{1}{2}(3x^2 - 1) \rangle = \int_{-1}^{1} \frac{1}{2}(3x^3 - x)\,dx = 0$ by symmetry.

More examples in function spaces

Example. For the inner product $\langle p, q\rangle = \int_{-1}^{1} p(x)q(x)\,dx$, find the norm of the function $p(x) = x$.

Solution. $\|x\| = \sqrt{\langle x, x\rangle} = \sqrt{\int_{-1}^{1} x^2\,dx} = \sqrt{2/3}$.

Example. For the same inner product, find the projection of the function $p(x) = 1$ on the function $q(x) = 3x^2$.

Solution. We want $\langle p, \frac{q}{\|q\|}\rangle \frac{q}{\|q\|}$ or $\frac{\langle p,q\rangle}{\langle q,q\rangle} q$, which equals

$$\frac{\int_{-1}^{1} 3x^2\,dx}{\int_{-1}^{1} 9x^4\,dx}\, q(x) = \frac{2}{18/5} 3x^2 = \frac{5}{3}x^2.$$

Normed linear spaces

Without an inner product, we cannot talk about angles or projections or the Pythagorean law or the Cauchy-Schwarz inequality, but we can still talk about distance if our vector space has a *norm*.

A norm must satisfy three properties:

- scaling: $\|c\mathbf{v}\| = |c|\,\|\mathbf{v}\|$ for every scalar c
- triangle inequality: $\|\mathbf{v}+\mathbf{w}\| \leq \|\mathbf{v}\| + \|\mathbf{w}\|$
- positivity: $\|\mathbf{v}\|$ is positive unless $\mathbf{v} = \mathbf{0}$

Examples of norms on R^2

- the usual Euclidean norm: $\|(x_1, x_2)^T\|_2 = \sqrt{x_1^2 + x_2^2}$
- the taxicab norm: $\|(x_1, x_2)^T\|_1 = |x_1| + |x_2|$
- the maximum norm: $\|(x_1, x_2)^T\|_\infty = \max(|x_1|, |x_2|)$