Math 304

Linear Algebra

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Highlights

From last time:

- inner products and norms

Today:

- orthonormal sets

Example in R^3

Let $\mathbf{u}_1 = \begin{pmatrix} \frac{1}{\sqrt{2}} \\ 0 \\ -\frac{1}{\sqrt{2}} \end{pmatrix}$, $\mathbf{u}_2 = \begin{pmatrix} \frac{1}{\sqrt{3}} \\ -\frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{3}} \end{pmatrix}$, and $\mathbf{u}_3 = \begin{pmatrix} \frac{1}{\sqrt{6}} \\ \frac{2}{\sqrt{6}} \\ \frac{1}{\sqrt{6}} \end{pmatrix}$.

These vectors form an *orthonormal set*: each vector is orthogonal to the others, and each vector has norm equal to 1.

The matrix U whose columns are $\mathbf{u}_1$, $\mathbf{u}_2$, and $\mathbf{u}_3$ is called an *orthogonal matrix*. Notice that $U^T U = I$ (the identity matrix), so $U^T = U^{-1}$. This property characterizes orthogonal matrices.

An orthogonal matrix preserves the scalar product: namely, $\langle U\mathbf{x}, U\mathbf{y}\rangle = \langle \mathbf{x}, \mathbf{y}\rangle$. Here is why:
$\langle U\mathbf{x}, U\mathbf{y}\rangle = (U\mathbf{x})^T U\mathbf{y} = \mathbf{x}^T U^T U\mathbf{y} = \mathbf{x}^T\mathbf{y} = \langle \mathbf{x}, \mathbf{y}\rangle$.
Therefore an orthogonal matrix also preserves the norm:
$\|U\mathbf{x}\| = \|\mathbf{x}\|$.

Example continued

Let $\mathbf{y} = 2\begin{pmatrix} \frac{1}{\sqrt{2}} \\ 0 \\ -\frac{1}{\sqrt{2}} \end{pmatrix} - 3\begin{pmatrix} \frac{1}{\sqrt{3}} \\ -\frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{3}} \end{pmatrix} + 4\begin{pmatrix} \frac{1}{\sqrt{6}} \\ \frac{2}{\sqrt{6}} \\ \frac{1}{\sqrt{6}} \end{pmatrix} = 2\mathbf{u}_1 - 3\mathbf{u}_2 + 4\mathbf{u}_3$.

Problem. Determine $\|\mathbf{y}\|$.

Solution. Since $\mathbf{y} = U\begin{pmatrix} 2 \\ -3 \\ 4 \end{pmatrix}$, and since multiplication by an orthogonal matrix preserves the norm,

$$\|\mathbf{y}\| = \left\| \begin{pmatrix} 2 \\ -3 \\ 4 \end{pmatrix} \right\| = \sqrt{4+9+16} = \sqrt{29}.$$

In general, $\|c_1\mathbf{u}_1 + c_2\mathbf{u}_2 + c_3\mathbf{u}_3\|^2 = c_1^2 + c_2^2 + c_3^2$.

The analogous statement for an arbitrary orthonormal set is *Parseval's formula*.

More on the example: Fourier coefficients

Problem. Express the vector $\mathbf{v} = (1, 2, 3)^T$ as a linear combination of $\mathbf{u}_1$, $\mathbf{u}_2$, and $\mathbf{u}_3$. In other words, find coefficients c_1, c_2, and c_3 such that $\mathbf{v} = c_1\mathbf{u}_1 + c_2\mathbf{u}_2 + c_3\mathbf{u}_3$ or

$$\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} = c_1 \begin{pmatrix} \frac{1}{\sqrt{2}} \\ 0 \\ -\frac{1}{\sqrt{2}} \end{pmatrix} + c_2 \begin{pmatrix} \frac{1}{\sqrt{3}} \\ -\frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{3}} \end{pmatrix} + c_3 \begin{pmatrix} \frac{1}{\sqrt{6}} \\ \frac{2}{\sqrt{6}} \\ \frac{1}{\sqrt{6}} \end{pmatrix}.$$

Solution. No row reduction necessary! Take the inner product with $\mathbf{u}_1$: thus $\langle \mathbf{v}, \mathbf{u}_1 \rangle = \langle c_1\mathbf{u}_1 + c_2\mathbf{u}_2 + c_3\mathbf{u}_3, \mathbf{u}_1 \rangle = c_1$ by orthonormality, so $c_1 = -2/\sqrt{2}$. Similarly, $c_2 = \langle \mathbf{v}, \mathbf{u}_2 \rangle = 2/\sqrt{3}$, and $c_3 = \langle \mathbf{v}, \mathbf{u}_3 \rangle = 8/\sqrt{6}$.

In general, any vector $\mathbf{v}$ can be represented in terms of an orthonormal basis $\mathbf{u}_1$, $\mathbf{u}_2$, and $\mathbf{u}_3$ via

$$\mathbf{v} = \langle \mathbf{v}, \mathbf{u}_1 \rangle \mathbf{u}_1 + \langle \mathbf{v}, \mathbf{u}_2 \rangle \mathbf{u}_2 + \langle \mathbf{v}, \mathbf{u}_3 \rangle \mathbf{u}_3.$$

Projection and approximation

Problem. Find the projection of the vector $\mathbf{v} = (2, 1, 1)^T$ onto the plane spanned by $\mathbf{u}_1$ and $\mathbf{u}_2$.

Solution. We can write

$$\mathbf{v} = \langle \mathbf{v}, \mathbf{u}_1 \rangle \mathbf{u}_1 + \langle \mathbf{v}, \mathbf{u}_2 \rangle \mathbf{u}_2 + \langle \mathbf{v}, \mathbf{u}_3 \rangle \mathbf{u}_3,$$

but $\mathbf{u}_3$ is orthogonal to the plane, so the projection equals

$$\langle \mathbf{v}, \mathbf{u}_1 \rangle \mathbf{u}_1 + \langle \mathbf{v}, \mathbf{u}_2 \rangle \mathbf{u}_2 = \begin{pmatrix} \frac{1}{2} \\ 0 \\ -\frac{1}{2} \end{pmatrix} + \begin{pmatrix} \frac{2}{3} \\ -\frac{2}{3} \\ \frac{2}{3} \end{pmatrix} = \begin{pmatrix} \frac{7}{6} \\ -\frac{2}{3} \\ \frac{1}{6} \end{pmatrix}.$$