

Linear Algebra

Write your **name**: Answer Key (2 points).

In **problems 1–5**, circle the correct answer. (5 points each)

1. Every linear system of three equations in four unknowns is consistent.
True False

Solution. False. In a consistent system $A\mathbf{x} = \mathbf{b}$, the vector \mathbf{b} has to be a linear combination of the columns of the matrix A . An example of an inconsistent system of three equations in four unknowns is

$$\begin{aligned}x_1 + x_2 + x_3 + x_4 &= 1 \\x_1 + x_2 + x_3 + x_4 &= 2 \\x_1 + x_2 + x_3 + x_4 &= 3.\end{aligned}$$

2. Three vectors in R^2 are always linearly dependent. True False

Solution. True. This is a particular case of Theorem 3.4.1 on page 147. Three vectors in a two-dimensional vector space are linearly dependent.

3. If A is a square matrix such that $\det(A) = 0$, then the homogeneous system $A\mathbf{x} = \mathbf{0}$ has infinitely many solutions. True False

Solution. True. This follows from (for example) Theorem 1.4.2 on page 65 or the subsequent Corollary 1.4.3.

4. If \mathbf{v}_1 , \mathbf{v}_2 , and \mathbf{v}_3 are elements of a vector space V , then the span of the vectors \mathbf{v}_1 , \mathbf{v}_2 , and \mathbf{v}_3 is a subspace of V . True False

Solution. True. This is the statement of Theorem 3.2.1 on page 128.

5. If A is a square matrix that is singular, then the matrix A^T (that is, the transpose of A) is singular too. True False

Solution. True. If A is singular, then $\det(A) = 0$. But $\det(A) = \det(A^T)$ (Theorem 2.1.2 on page 96), so $\det(A^T) = 0$, and hence A^T is singular.

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In **problems 6–9**, fill in the blanks. (7 points per problem)

$$6. \begin{pmatrix} 3 & 5 & 1 \\ -2 & 0 & 2 \end{pmatrix} \begin{pmatrix} 2 & 1 \\ 1 & \boxed{3} \\ 4 & 1 \end{pmatrix} = \begin{pmatrix} 15 & 19 \\ \boxed{4} & 0 \end{pmatrix}$$

$$7. \det \begin{pmatrix} 2 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 1 & 6 & 2 & 0 \\ 1 & 1 & \boxed{8} & 3 \end{pmatrix} = 18$$

$$8. \text{ If } \left(\begin{array}{cccc|c} 1 & 0 & 0 & 0 & 5 \\ 0 & 1 & 0 & 0 & 4 \\ 0 & 0 & 1 & 0 & 3 \\ 0 & 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 & 0 \end{array} \right) \text{ is the end stage of the Gauss-Jordan reduction}$$

algorithm applied to the augmented matrix of a system of linear equations, then the system of equations has **exactly one** solution(s). [none, exactly one, or infinitely many?]

9. Vectors $\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_n$ form a *basis* for the vector space V if and only if $\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_n$ are linearly independent and $\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_n$ span V .

In **problems 10–12**, show your work and explain your method. Continue on the back if you need more space. (15 points each)

10. If $A = \begin{pmatrix} 1 & 6 & 3 \\ 3 & 24 & 9 \\ 0 & 24 & 9 \end{pmatrix}$, find a lower triangular matrix L and an upper triangular matrix U such that $A = LU$.

Solution. Row operations will determine the factorization:

$$\begin{pmatrix} 1 & 6 & 3 \\ 3 & 24 & 9 \\ 0 & 24 & 9 \end{pmatrix} \xrightarrow{R_2-3R_1} \begin{pmatrix} 1 & 6 & 3 \\ 0 & 6 & 0 \\ 0 & 24 & 9 \end{pmatrix} \xrightarrow{R_3-4R_2} \begin{pmatrix} 1 & 6 & 3 \\ 0 & 6 & 0 \\ 0 & 0 & 9 \end{pmatrix}.$$

$$\text{Then } L = \begin{pmatrix} 1 & 0 & 0 \\ 3 & 1 & 0 \\ 0 & 4 & 1 \end{pmatrix} \text{ and } U = \begin{pmatrix} 1 & 6 & 3 \\ 0 & 6 & 0 \\ 0 & 0 & 9 \end{pmatrix}.$$

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11. Can every polynomial of degree 2 or less be written as a linear combination of the three polynomials $1 + x - x^2$, $1 - x$, and $1 + x^2$? Explain why or why not.

Solution. Yes. One can see why in several different ways.

Method 1. We want to know if, for every choice of numbers a , b , and c , there are numbers c_1 , c_2 , and c_3 such that

$$c_1(1 + x - x^2) + c_2(1 - x) + c_3(1 + x^2) = ax^2 + bx + c.$$

Matching powers of x shows that we want to solve:

$$\begin{array}{l} -c_1 + c_3 = a \\ c_1 - c_2 = b \\ c_1 + c_2 + c_3 = c \end{array} \quad \text{or} \quad \begin{pmatrix} -1 & 0 & 1 \\ 1 & -1 & 0 \\ 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix} = \begin{pmatrix} a \\ b \\ c \end{pmatrix}.$$

The determinant of the matrix equals 3. The matrix is invertible since the determinant is not zero, so we can solve the linear system for arbitrary values of a , b , and c .

Method 2. It is enough to show that the standard basis elements 1 , x , and x^2 can be written as linear combinations of the three given polynomials. The sum of the three given polynomials equals 3, so the constant polynomial 1 is in their span. Then $1 - (1 - x) = x$ is in the span too, and so is $(1 + x^2) - 1 = x^2$.

Method 3. Since we are working in a three-dimensional vector space, the three given elements will span the space if they are linearly independent. We can verify linear independence by showing that the Wronskian determinant is not equal to 0:

$$\begin{aligned} & \begin{vmatrix} 1 + x - x^2 & 1 - x & 1 + x^2 \\ 1 - 2x & -1 & 2x \\ -2 & 0 & 2 \end{vmatrix} \stackrel{C_1+C_3}{=} \begin{vmatrix} 2 + x & 1 - x & 1 + x^2 \\ 1 & -1 & 2x \\ 0 & 0 & 2 \end{vmatrix} \\ & = 2 \begin{vmatrix} 2 + x & 1 - x \\ 1 & -1 \end{vmatrix} \stackrel{C_1+C_2}{=} 2 \begin{vmatrix} 3 & 1 - x \\ 0 & -1 \end{vmatrix} = -6 \neq 0. \end{aligned}$$

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12. Write down the 2×3 matrix whose first row is the first three digits of your student identification number and whose second row is the last three digits of your student identification number. Then determine the nullspace of your matrix.

Solution. The answer depends on your identification number. Here is a solution for a fictitious student with ID number 721004568.

$$\begin{aligned} & \left(\begin{array}{ccc|c} 7 & 2 & 1 & 0 \\ 5 & 6 & 8 & 0 \end{array} \right) \xrightarrow{R_1/7} \left(\begin{array}{ccc|c} 1 & \frac{2}{7} & \frac{1}{7} & 0 \\ 5 & 6 & 8 & 0 \end{array} \right) \xrightarrow{R_2-5R_1} \left(\begin{array}{ccc|c} 1 & \frac{2}{7} & \frac{1}{7} & 0 \\ 0 & \frac{32}{7} & \frac{51}{7} & 0 \end{array} \right) \\ & \xrightarrow{R_2/(32/7)} \left(\begin{array}{ccc|c} 1 & \frac{2}{7} & \frac{1}{7} & 0 \\ 0 & 1 & \frac{51}{32} & 0 \end{array} \right) \xrightarrow{R_1-\frac{2}{7}R_2} \left(\begin{array}{ccc|c} 1 & 0 & -\frac{5}{16} & 0 \\ 0 & 1 & \frac{51}{32} & 0 \end{array} \right). \end{aligned}$$

Therefore the nullspace consists of vectors $\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$ for which x_3 is arbitrary, $x_2 = -\frac{51}{32}x_3$, and $x_1 = \frac{5}{16}x_3$. Setting $x_3 = 32\alpha$, one can describe the nullspace as the set of vectors of the form $\begin{pmatrix} 10\alpha \\ -51\alpha \\ 32\alpha \end{pmatrix}$, where α is ar-

bitrary. In other words, the nullspace is the span of the vector $\begin{pmatrix} 10 \\ -51 \\ 32 \end{pmatrix}$.

Remark. One can avoid fractions by doing the computation in the following tricky way:

$$\left(\begin{array}{ccc|c} 7 & 2 & 1 & 0 \\ 5 & 6 & 8 & 0 \end{array} \right) \xrightarrow[\begin{smallmatrix} R_1 \times 5 \\ R_2 \times 7 \end{smallmatrix}]{R_1 \times 5} \left(\begin{array}{ccc|c} 35 & 10 & 5 & 0 \\ 35 & 42 & 56 & 0 \end{array} \right) \xrightarrow[\begin{smallmatrix} R_1/5 \\ R_2-R_1 \end{smallmatrix}]{R_2-R_1} \left(\begin{array}{ccc|c} 7 & 2 & 1 & 0 \\ 0 & 32 & 51 & 0 \end{array} \right),$$

and now set $x_3 = 32\alpha$. Back substitution gives $x_2 = -51\alpha$ and then $x_1 = 10\alpha$. This is the same answer as before.

Incidentally, there is an easy way to do this problem by using a tool from another course. Namely, the vector cross product of the two rows of the matrix is a vector orthogonal to both rows, and hence that vector is in the null space of the matrix. Since the two rows are linearly independent, there is only one direction in R^3 orthogonal to both rows, so the cross product vector spans the nullspace of the matrix.