Linear Algebra

1. Suppose $A = \begin{pmatrix} 1 & 2 & 0 & 0 \\ 0 & 3 & -3 & 6 \\ 1 & 0 & 2 & -4 \\ 0 & 1 & -1 & 2 \end{pmatrix}$. Find a basis for $R(A)^{\perp}$.

Solution. An equivalent problem is to find the nullspace of the transpose A^T. Row reducing A^T leads to the matrix $\begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 3 & -2 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$, so the vectors $\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix}$ in the nullspace of A^T have the form $\begin{pmatrix} -x_3 \\ \frac{2}{3}x_3 - \frac{1}{3}x_4 \\ x_3 \\ x_4 \end{pmatrix}$, or $x_3 \begin{pmatrix} -1 \\ 2/3 \\ 1 \\ 0 \end{pmatrix} + x_4 \begin{pmatrix} 0 \\ -1/3 \\ 0 \\ 1 \end{pmatrix}$.

Therefore the two vectors $\begin{pmatrix} -1 \\ 2/3 \\ 1 \\ 0 \end{pmatrix}$ and $\begin{pmatrix} 0 \\ -1/3 \\ 0 \\ 1 \end{pmatrix}$ form a basis for the nullspace of A^T. An alternative answer without fractions is the pair of vectors $\begin{pmatrix} -3 \\ 2 \\ 3 \\ 0 \end{pmatrix}$ and $\begin{pmatrix} 0 \\ -1 \\ 0 \\ 3 \end{pmatrix}$.

You can check that these vectors are indeed orthogonal to the columns of the matrix A.

Linear Algebra

2. When asked for a least-squares solution to the linear system

$$\begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \\ -1 \end{pmatrix},$$

MATLAB returns the solution $(x_1, x_2, x_3) = (0, 1, \frac{1}{2})$, but Maple returns the solution $(x_1, x_2, x_3) = (\frac{1}{4}, 1, \frac{1}{4})$. Explain the discrepancy.

Solution. The matrix does not have maximal rank (indeed, the first and third columns are linearly dependent), so the least-squares problem does not have a unique solution. There are infinitely many vectors $\mathbf{x}$ that minimize the length of the difference $A\mathbf{x} - \mathbf{b}$, where $A = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix}$ and $\mathbf{b} = \begin{pmatrix} 2 \\ 1 \\ -1 \end{pmatrix}$.

Since $\begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \\ \frac{1}{2} \end{pmatrix} = \begin{pmatrix} \frac{1}{2} \\ 1 \\ \frac{1}{2} \end{pmatrix} = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix} \begin{pmatrix} \frac{1}{4} \\ 1 \\ \frac{1}{4} \end{pmatrix}$, MATLAB's solution miminizes the norm of $A\mathbf{x} - \mathbf{b}$ if and only if Maple's solution does.

The least-squares problem $A^T A\mathbf{x} = A^T\mathbf{b}$ in this example becomes the problem

$$\begin{pmatrix} 2 & 0 & 2 \\ 0 & 1 & 0 \\ 2 & 0 & 2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix},$$

and row reducing shows that the solutions to the least-squares problem have the form $\begin{pmatrix} \frac{1}{2} - x_3 \\ 1 \\ x_3 \end{pmatrix}$ with x_3 arbitrary. MATLAB's solution corresponds to the value $x_3 = 1/2$, and Maple's solution corresponds to the value $x_3 = 1/4$.