Linear Algebra

1. Suppose $A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \end{pmatrix}$ and $B = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$. Compute whichever of the matrix products AB and BA makes sense.

   Solution. Since A is a 2×3 matrix and B is a 2×2 matrix, the product that makes sense is BA. If you know how matrix multiplication works, then you can write down the answer by inspection; but here is a thematic way to see what the answer is. Since $B\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 3 \end{pmatrix}$ and $B\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 2 \\ 4 \end{pmatrix}$, and matrix multiplication acts linearly on the columns of the second matrix, it follows that $BA = \begin{pmatrix} 1 & 2 & 2 \\ 3 & 4 & 4 \end{pmatrix}$.

2. Express the vector $\begin{pmatrix} -4 \\ 12 \\ 11 \end{pmatrix}$ as a linear combination of the three vectors $\begin{pmatrix} 1 \\ 6 \\ 3 \end{pmatrix}$, $\begin{pmatrix} 3 \\ 0 \\ 4 \end{pmatrix}$, and $\begin{pmatrix} 5 \\ 3 \\ 1 \end{pmatrix}$.

   Solution. Set up the system $x_1 \begin{pmatrix} 1 \\ 6 \\ 3 \end{pmatrix} + x_2 \begin{pmatrix} 3 \\ 0 \\ 4 \end{pmatrix} + x_3 \begin{pmatrix} 5 \\ 3 \\ 1 \end{pmatrix} = \begin{pmatrix} -4 \\ 12 \\ 11 \end{pmatrix}$ and solve using Gaussian elimination. A row-echelon form of the augmented matrix $\left(\begin{array}{ccc|c} 1 & 3 & 5 & -4 \\ 6 & 0 & 3 & 12 \\ 3 & 4 & 1 & 11 \end{array}\right)$ is $\left(\begin{array}{ccc|c} 1 & 3 & 5 & -4 \\ 0 & 1 & \frac{3}{2} & -2 \\ 0 & 0 & 1 & -2 \end{array}\right)$ [add -6 times row 1 to row 2; add -3 times row 1 to row 3; divide row 2 by -18; and add 5 times row 2 to row 3]. Back substitution gives $x_3 = -2$, $x_2 = 1$, and $x_1 = 3$. Thus

$$\begin{pmatrix} -4 \\ 12 \\ 11 \end{pmatrix} = 3 \begin{pmatrix} 1 \\ 6 \\ 3 \end{pmatrix} + \begin{pmatrix} 3 \\ 0 \\ 4 \end{pmatrix} - 2 \begin{pmatrix} 5 \\ 3 \\ 1 \end{pmatrix}.$$