Linear Algebra

1. Find the value of a for which $\det \begin{pmatrix} 1 & 2 & 1 & -3 \\ -1 & -1 & 2 & 2 \\ -2 & -4 & -4 & -1 \\ -3 & -6 & -9 & a \end{pmatrix} = 0.$

   Solution. Use row operations $R_2 + R_1$, $R_3 + 2R_1$, and $R_4 + 3R_1$ to get

   $$0 = \begin{vmatrix} 1 & 2 & 1 & -3 \\ 0 & 1 & 3 & -1 \\ 0 & 0 & -2 & -7 \\ 0 & 0 & -6 & a-9 \end{vmatrix} \overset{R_4 - 3R_3}{=} \begin{vmatrix} 1 & 2 & 1 & -3 \\ 0 & 1 & 3 & -1 \\ 0 & 0 & -2 & -7 \\ 0 & 0 & 0 & a+12 \end{vmatrix}$$

   (since these row operations do not change the value of the determinant). The determinant of an upper triangular matrix equals the product of the entries on the main diagonal, so we find that $0 = -2(a + 12)$, or $a = -12$.

2. If $A = \begin{pmatrix} 1 & 3 & -4 \\ 2 & -1 & -1 \\ -1 & -3 & 4 \end{pmatrix}$, find the nullspace $N(A)$, that is, find the set of solutions to the homogeneous equation $A\mathbf{x} = \mathbf{0}$.
   (This is exercise 4c on page 132.)

   Solution. Use Gauss-Jordan reduction to bring the augmented matrix to reduced row-echelon form:

   $$\left(\begin{array}{rrr|r} 1 & 3 & -4 & 0 \\ 2 & -1 & -1 & 0 \\ -1 & -3 & 4 & 0 \end{array}\right) \xrightarrow[R_3+R_1]{R_2-2R_1} \left(\begin{array}{rrr|r} 1 & 3 & -4 & 0 \\ 0 & -7 & 7 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

   $$\xrightarrow{R_2/(-7)} \left(\begin{array}{rrr|r} 1 & 3 & -4 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right) \xrightarrow{R_1-3R_2} \left(\begin{array}{rrr|r} 1 & 0 & -1 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right).$$

   The free variable x_3 is arbitrary, $x_2 = x_3$, and $x_1 = x_3$. Thus $N(A)$ is spanned by the vector $\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$. In other words, $N(A)$ consists of all vectors of the form $\begin{pmatrix} \alpha \\ \alpha \\ \alpha \end{pmatrix}$, where α is an arbitrary real number.