Linear Algebra

1. Suppose E is the basis $\left[\begin{pmatrix}5\\3\end{pmatrix}, \begin{pmatrix}3\\2\end{pmatrix}\right]$ of R^2, and $\mathbf{z} = \begin{pmatrix}10\\7\end{pmatrix}$. Find the E-basis coordinates of the vector $\mathbf{z}$. [exercise 4, page 161]

   Solution. The matrix $\begin{pmatrix}5 & 3\\3 & 2\end{pmatrix}$ whose columns are the E-basis vectors is the transition matrix from the E-basis to the standard basis. The inverse matrix $\begin{pmatrix}2 & -3\\-3 & 5\end{pmatrix}$ is the transition matrix from standard coordinates to E-basis coordinates. So the E-coordinates of $\mathbf{z}$ are $\begin{pmatrix}2 & -3\\-3 & 5\end{pmatrix}\begin{pmatrix}10\\7\end{pmatrix} = \begin{pmatrix}-1\\5\end{pmatrix}$. This means $\mathbf{z} = -\begin{pmatrix}5\\3\end{pmatrix} + 5\begin{pmatrix}3\\2\end{pmatrix}$.

2. If $A = \begin{pmatrix}1 & 0 & 9 & -3\\2 & 1 & 8 & -1\\3 & 2 & 7 & 1\\4 & 3 & 6 & 3\end{pmatrix}$, find a basis for the column space of A.

   Solution. Row reduce ($R_2 - 2R_1$, $R_3 - 3R_1$, $R_4 - 4R_1$; then $R_3 - 2R_2$, $R_4 - 3R_2$) to get $\begin{pmatrix}1 & 0 & 9 & -3\\0 & 1 & -10 & 5\\0 & 0 & 0 & 0\\0 & 0 & 0 & 0\end{pmatrix}$. From this echelon form one sees that the rank (hence the dimension of the column space) equals 2, and the first two columns are linearly independent. Hence the vectors $\begin{pmatrix}1\\2\\3\\4\end{pmatrix}$ and $\begin{pmatrix}0\\1\\2\\3\end{pmatrix}$ are a basis for the column space. The answer is not unique. One could use any other pair of linearly independent vectors with the same span—for instance, the vectors $\begin{pmatrix}0\\1\\2\\3\end{pmatrix}$ and $\begin{pmatrix}1\\0\\-1\\-2\end{pmatrix}$.