Linear Algebra

This worksheet concerns matrix representations of linear transformations. Suppose L is a linear operator on R^3 defined by the formula

$$L \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 2x_1 - x_2 + x_3 \\ 2x_1 - x_2 + x_3 \\ -6x_2 + 6x_3 \end{pmatrix}.$$

1. Find the matrix A that represents the transformation L with respect to the standard basis.

2. Determine the rank of the matrix A, and find a basis for the nullspace of the matrix A.

3. Is the transformation L one-to-one? Is it onto? How do you know?

4. Let $\mathbf{u}_1 = \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix}$, $\mathbf{u}_2 = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$, and $\mathbf{u}_3 = \begin{pmatrix} 1 \\ 1 \\ 3 \end{pmatrix}$. One of these three vectors should look familiar from a previous part of the worksheet. Which vector do you recognize, and why?

5. Find the transition matrix U from $\mathbf{u}$-coordinates to standard coordinates, and find the transition matrix U^{-1} from standard coordinates to $\mathbf{u}$-coordinates.

6. Find the matrix B that represents the transformation L with respect to the $\mathbf{u}$-basis as follows: use the matrix U to go from $\mathbf{u}$-coordinates to standard coordinates, then use the matrix A to execute the transformation L in standard coordinates, then use the matrix U^{-1} to go back to the $\mathbf{u}$-coordinates.

7. Why is the $\mathbf{u}$-basis a particularly nice basis for describing the transformation L?

8. Compute $L^{615}(\mathbf{u}_1)$.

9. The matrix A and the matrix $B = U^{-1}AU$ are called *similar matrices*. Show that A^2 and B^2 are similar matrices.

10. What is the relation between the determinant of A and the determinant of B? What is the relation between the trace of A and the trace of B?