Linear Algebra

Let $A = \begin{pmatrix} 4 & 2 & 1 \\ 2 & 0 & 1 \\ 2 & 0 & -1 \\ 1 & 2 & 1 \end{pmatrix}$. [exercise 8, page 282]

1. Find an orthonormal basis for the column space of A by the following procedure, called "the Gram-Schmidt process".

   (a) Divide the first column by a suitable constant to make the column have norm 1.

   (b) Subtract from the second column its projection onto the (new) first column.

   (c) Normalize the new second column to have length 1.

   (d) Subtract from the third column its projection onto the subspace spanned by the first two columns.

   (e) Normalize the new third column.

   Verify that your final matrix does indeed have orthonormal columns.

2. For each of the above steps (a)–(e), find an upper triangular elementary matrix that implements that step via matrix multiplication on the *right*.

3. You now have a matrix Q with orthonormal columns and elementary matrices E_1, E_2, E_3, E_4, and E_5 such that $AE_1E_2E_3E_4E_5 = Q$. Deduce that $A = QR$, where $R = E_5^{-1}E_4^{-1}E_3^{-1}E_2^{-1}E_1^{-1}$.

4. The matrix R is an invertible upper triangular matrix since it is a product of invertible upper triangular matrices. Compute R.

5. Let $\mathbf{b} = \begin{pmatrix} 0 \\ 1 \\ 1 \\ 1 \end{pmatrix}$. Show that finding the least-squares solution to $A\mathbf{x} = \mathbf{b}$ reduces to solving the equation $R\mathbf{x} = Q^T\mathbf{b}$, and then solve this new equation by back substitution.