# Linear Algebra

Write your **name**: Answer Key (2 points).

In **problems 1–5**, circle the correct answer. (5 points each)

1. If $A$ is a $3 \times 3$ matrix, then $A$ is a singular matrix if and only if the linear system $A\mathbf{x} = \mathbf{0}$ is inconsistent. `True` `False`

   **Solution.** The statement is false, because the *homogeneous* system $A\mathbf{x} = 0$ is *always* consistent (since $\mathbf{x} = 0$ is a solution).

2. If $A$ is an invertible $3 \times 3$ matrix, then $(A^T)^{-1} = (A^{-1})^T$.
   `True` `False`

   **Solution.** The statement is true, and here is one way to see why. Since $A^{-1}A = I$, taking the transpose shows that $A^T(A^{-1})^T = I$. By the definition of inverse matrix, this equation means that $(A^T)^{-1} = (A^{-1})^T$.

3. If $\mathbf{v}_1$, $\mathbf{v}_2$, and $\mathbf{v}_3$ are elements of a vector space $V$, then the span of $\mathbf{v}_1$, $\mathbf{v}_2$, and $\mathbf{v}_3$ (that is, the set of all linear combinations of $\mathbf{v}_1$, $\mathbf{v}_2$, and $\mathbf{v}_3$) is a subspace of $V$. `True` `False`

   **Solution.** True; this is one of the fundamental ways of creating subspaces. The statement is Theorem 3.2.1 on page 128 of the textbook.

4. A linear system $A\mathbf{x} = \mathbf{b}$ is consistent if and only if the column vector $\mathbf{b}$ can be written as a linear combination of the column vectors of the matrix $A$. `True` `False`

   **Solution.** True, because the matrix product $A\mathbf{x}$ can be interpreted as a linear combination of the columns of the matrix. The statement is Theorem 1.3.1 on page 37 of the textbook.

5. The polynomials $1 + x$, $1 + x^2$, and $2 + x + x^2$ form a basis for the three-dimensional vector space $P_3$ (the vector space of polynomials of degree less than 3). `True` `False`

   **Solution.** False: since $(1 + x) + (1 + x^2) = 2 + x + x^2$, the three given polynomials are linearly dependent, so they cannot constitute a basis.

# Linear Algebra

In **problems 6–9**, fill in the blanks. (7 points per problem)

6. If $A = \begin{pmatrix} \boxed{\phantom{0}} & 0 \\ 0 & 4 \end{pmatrix}$ and $B = \begin{pmatrix} 1 & -1 \\ 5 & \boxed{\phantom{0}} \end{pmatrix}$, then $2A+B = \begin{pmatrix} 7 & \boxed{\phantom{0}} \\ 5 & 2 \end{pmatrix}$.

**Solution.** If $A = \begin{pmatrix} \boxed{3} & 0 \\ 0 & 4 \end{pmatrix}$ and $B = \begin{pmatrix} 1 & -1 \\ 5 & \boxed{-6} \end{pmatrix}$, then $2A+B = \begin{pmatrix} 7 & \boxed{-1} \\ 5 & 2 \end{pmatrix}$.

7. $\begin{pmatrix} 1 & 0 & 0 \\ 3 & 1 & 0 \\ 0 & 0 & \boxed{\phantom{0}} \end{pmatrix}^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ \boxed{\phantom{0}} & 1 & 0 \\ 0 & 0 & 1/4 \end{pmatrix}$

**Solution.** $\begin{pmatrix} 1 & 0 & 0 \\ 3 & 1 & 0 \\ 0 & 0 & \boxed{4} \end{pmatrix}^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ \boxed{-3} & 1 & 0 \\ 0 & 0 & 1/4 \end{pmatrix}$

8. If $A$ is an $n \times m$ matrix, then the set of all solutions to the homogeneous system $A\mathbf{x} = \mathbf{0}$ is called the ____________________ of $A$.

**Solution.** If $A$ is an $n \times m$ matrix, then the set of all solutions to the homogeneous system $A\mathbf{x} = \mathbf{0}$ is called the nullspace of $A$.

9. If $\left(\begin{array}{cccc|c} 1 & 2 & 0 & 5 & 4 \\ 0 & 0 & 1 & 3 & 0 \end{array}\right)$ is the end stage of the Gauss-Jordan reduction algorithm applied to the augmented matrix of a system of linear equations, then the system of equations has ____________________ solution(s). [none, exactly one, or infinitely many?]

**Solution.** The system is consistent and has two free variables, so there are infinitely many solutions.

# Linear Algebra

In **problems 10–12**, show your work and explain your method. Continue on the back if you need more space. (15 points each)

10. If $A = \begin{pmatrix} 1 & 3 & 0 \\ 6 & 19 & 4 \\ 0 & 8 & 33 \end{pmatrix}$, find a lower triangular matrix $L$ and an upper triangular matrix $U$ such that $A = LU$.

**Solution.** Let $E_1$ be the elementary matrix $\begin{pmatrix} 1 & 0 & 0 \\ -6 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$. Then $E_1 A = \begin{pmatrix} 1 & 3 & 0 \\ 0 & 1 & 4 \\ 0 & 8 & 33 \end{pmatrix}$. Let $E_2$ be the elementary matrix $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -8 & 1 \end{pmatrix}$. Then $E_2 E_1 A = \begin{pmatrix} 1 & 3 & 0 \\ 0 & 1 & 4 \\ 0 & 0 & 1 \end{pmatrix} := U$. We have $A = LU$ if

$$L = E_1^{-1} E_2^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ 6 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 8 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 6 & 1 & 0 \\ 0 & 8 & 1 \end{pmatrix}.$$

11. Determine the value of $a$ for which

$$\det \begin{pmatrix} 3 & 0 & 4 & 0 \\ 0 & 3 & 0 & 4 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & a \end{pmatrix} = 0.$$

**Solution.** One can either compute the determinant using a cofactor expansion or simplify the determinant using row operations. Here is a solution via the second method.

Making two row interchanges (hence two cancelling sign changes) gives

$$\begin{vmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & a \\ 3 & 0 & 4 & 0 \\ 0 & 3 & 0 & 4 \end{vmatrix}.$$

Now subtracting 3 times the first row from the third row and 3 times the second row from the fourth row gives

$$\begin{vmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & a \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 4-3a \end{vmatrix}.$$

The determinant of this triangular matrix is the product of the entries on the main diagonal, $4 - 3a$. Therefore the determinant is equal to 0 when $a = 4/3$.

12. Let $\mathbf{v}_1$ be the vector in $R^3$ whose entries are the first three digits of your student identification number. Similarly, let $\mathbf{v}_2$ be the vector whose entries are the middle three digits of your identification number, and let $\mathbf{v}_3$ be the vector whose entries are the last three digits of your identification number. Are your vectors $\mathbf{v}_1$, $\mathbf{v}_2$, and $\mathbf{v}_3$ linearly independent vectors in $R^3$? Explain why or why not.

    **Solution.** One method is to write the three vectors as the columns of a matrix and to compute the determinant. If the determinant is equal to 0, then the vectors are linearly dependent; otherwise the vectors are linearly independent (Theorem 3.3.1 on page 139 of the textbook). Alternatively, you could reduce the matrix to echelon form. If the echelon form has a row of zeroes at the bottom, then the original vectors are linearly dependent; otherwise the vectors are linearly independent.