Linear Algebra

1. Let $A = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix}$ and $\mathbf{b} = \begin{pmatrix} 10 \\ 11 \\ 12 \end{pmatrix}$. Is the vector $\mathbf{b}$ in the column space of the matrix A? Explain why or why not.

Solution. An equivalent question is whether the system $A\mathbf{x} = \mathbf{b}$ is consistent. That question can be answered by Gaussian elimination applied to an augmented matrix:

$$\left(\begin{array}{ccc|c} 1 & 2 & 3 & 10 \\ 4 & 5 & 6 & 11 \\ 7 & 8 & 9 & 12 \end{array}\right) \xrightarrow[R_3 \to R_3 - 7R_1]{R_2 \to R_2 - 4R_1} \left(\begin{array}{ccc|c} 1 & 2 & 3 & 10 \\ 0 & -3 & -6 & -29 \\ 0 & -6 & -12 & -58 \end{array}\right)$$

$$\xrightarrow{R_3 \to R_3 - 2R_2} \left(\begin{array}{ccc|c} 1 & 2 & 3 & 10 \\ 0 & -3 & -6 & -29 \\ 0 & 0 & 0 & 0 \end{array}\right).$$

The echelon form shows that the system is consistent; hence the vector $\mathbf{b}$ does belong to the column space of the matrix A.

You were not required to write down a specific linear combination of the columns that equals $\mathbf{b}$, but the echelon form shows how to do so. Namely, x_3 is a free variable, $x_2 = \frac{29}{3} - 2x_3$, and back substitution shows that $x_1 = -\frac{28}{3} + x_3$. Taking $x_3 = 1/3$ is particularly convenient, since then x_1 and x_2 become integers; we then have the following representation of the vector $\mathbf{b}$:

$$-9 \begin{pmatrix} 1 \\ 4 \\ 7 \end{pmatrix} + 9 \begin{pmatrix} 2 \\ 5 \\ 8 \end{pmatrix} + \frac{1}{3} \begin{pmatrix} 3 \\ 6 \\ 9 \end{pmatrix} = \begin{pmatrix} 10 \\ 11 \\ 12 \end{pmatrix}.$$

Alternatively, you might set x_3 equal to 0, which leads to the following representation of $\mathbf{b}$:

$$-\frac{28}{3} \begin{pmatrix} 1 \\ 4 \\ 7 \end{pmatrix} + \frac{29}{3} \begin{pmatrix} 2 \\ 5 \\ 8 \end{pmatrix} + 0 \begin{pmatrix} 3 \\ 6 \\ 9 \end{pmatrix} = \begin{pmatrix} 10 \\ 11 \\ 12 \end{pmatrix}.$$

Linear Algebra

2. Let L be the linear transformation from R^3 into R^2 such that (with respect to the standard basis) $L\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} x_2 \\ x_1 \end{pmatrix}$. If $\mathbf{u}_1 = \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}$, $\mathbf{u}_2 = \begin{pmatrix} 3 \\ 4 \\ -4 \end{pmatrix}$, $\mathbf{u}_3 = \begin{pmatrix} 1 \\ 1 \\ -2 \end{pmatrix}$, $\mathbf{v}_1 = \begin{pmatrix} 3 \\ 2 \end{pmatrix}$, and $\mathbf{v}_2 = \begin{pmatrix} 4 \\ 3 \end{pmatrix}$, find the matrix representation of L with respect to the ordered bases $[\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3]$ and $[\mathbf{v}_1, \mathbf{v}_2]$.

Solution. Since $L(\mathbf{u}_1) = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$, $L(\mathbf{u}_2) = \begin{pmatrix} 4 \\ 3 \end{pmatrix}$, and $L(\mathbf{u}_3) = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$, the matrix $\begin{pmatrix} 2 & 4 & 1 \\ 1 & 3 & 1 \end{pmatrix}$ represents L with respect to the basis $[\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3]$ in R^3 and the standard basis in R^2. The matrix $\begin{pmatrix} 3 & 4 \\ 2 & 3 \end{pmatrix}$ is the transition matrix from the basis $[\mathbf{v}_1, \mathbf{v}_2]$ to the standard basis in R^2, and the inverse matrix $\begin{pmatrix} 3 & -4 \\ -2 & 3 \end{pmatrix}$ is the transition matrix from the standard basis to the basis $[\mathbf{v}_1, \mathbf{v}_2]$. Therefore the product matrix

$$\begin{pmatrix} 3 & -4 \\ -2 & 3 \end{pmatrix} \begin{pmatrix} 2 & 4 & 1 \\ 1 & 3 & 1 \end{pmatrix} = \begin{pmatrix} 2 & 0 & -1 \\ -1 & 1 & 1 \end{pmatrix}$$

is the required matrix that represents L with respect to the bases $[\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3]$ and $[\mathbf{v}_1, \mathbf{v}_2]$.