Linear Algebra

1. If $\mathbf{x} = \begin{pmatrix} 2 \\ -5 \\ 4 \end{pmatrix}$ and $\mathbf{y} = \begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix}$, find the vector projection of $\mathbf{x}$ onto $\mathbf{y}$.
[This is exercise 3(d) on page 224 of the textbook.]

Solution. The vector projection equals the scalar projection $\mathbf{x} \cdot \frac{\mathbf{y}}{\|\mathbf{y}\|}$ times a unit vector $\frac{\mathbf{y}}{\|\mathbf{y}\|}$ in the direction of $\mathbf{y}$, or

$$\left(\frac{\mathbf{x}\cdot\mathbf{y}}{\|\mathbf{y}\|^2}\right)\mathbf{y} = \left(\frac{2-10-4}{1+4+1}\right)\mathbf{y} = -2\mathbf{y} = \begin{pmatrix} -2 \\ -4 \\ 2 \end{pmatrix}.$$

2. Let L be the linear transformation from R^2 to R^2 given with respect to the standard basis by $L\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ x_2 \end{pmatrix}$, let $\mathbf{u}_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$, and let $\mathbf{u}_2 = \begin{pmatrix} -1 \\ 1 \end{pmatrix}$. Find the matrix that represents the transformation L with respect to the basis $[\mathbf{u}_1, \mathbf{u}_2]$.
[This is exercise 1(e) on page 204 of the textbook.]

Solution. We know two ways to solve this problem.

Method 1 Since $L(\mathbf{u}_1) = \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \frac{1}{2}\mathbf{u}_1 + \frac{1}{2}\mathbf{u}_2$, the first column of the matrix representing L with respect to the basis $[\mathbf{u}_1, \mathbf{u}_2]$ is $\begin{pmatrix} \frac{1}{2} \\ \frac{1}{2} \end{pmatrix}$. Similarly, $L(\mathbf{u}_2) = \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \frac{1}{2}\mathbf{u}_1 + \frac{1}{2}\mathbf{u}_2$, so the second column of the matrix is $\begin{pmatrix} \frac{1}{2} \\ \frac{1}{2} \end{pmatrix}$. Thus the required matrix is $\begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix}$.

Method 2 The matrix A representing L with respect to the standard basis is evidently $\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$. The matrix we seek is $S^{-1}AS$, where S is the transition matrix from the basis $[\mathbf{u}_1, \mathbf{u}_2]$ to the standard basis. This transition matrix is $\begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}$ (the columns are $\mathbf{u}_1$ and $\mathbf{u}_2$), and a simple computation shows that $S^{-1} = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2} \end{pmatrix}$. Thus

$$S^{-1}AS = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2} \end{pmatrix} \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix},$$

the same answer as before.