Linear Algebra

Instructions Please write your name in the upper right-hand corner of the page. Use complete sentences, along with any necessary supporting calculations, to answer the following questions.

1. Find a basis for the column space of the matrix $\begin{pmatrix} 1 & 1 & 1 & 5 \\ 2 & 2 & 1 & 8 \\ 3 & 3 & 2 & 13 \end{pmatrix}$.

   Solution. Use Gaussian elimination:

$$\begin{pmatrix} 1 & 1 & 1 & 5 \\ 2 & 2 & 1 & 8 \\ 3 & 3 & 2 & 13 \end{pmatrix} \xrightarrow[R3 \to R3 - 3R1]{R2 \to R2 - 2R1} \begin{pmatrix} 1 & 1 & 1 & 5 \\ 0 & 0 & -1 & -2 \\ 0 & 0 & -1 & -2 \end{pmatrix}$$

$$\xrightarrow[R2 \to -R2]{R3 \to R3 - R2} \begin{pmatrix} 1 & 1 & 1 & 5 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 \end{pmatrix} \xrightarrow{R1 \to R1 - R2} \begin{pmatrix} 1 & 1 & 0 & 3 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

   The conclusion from the calculation is that columns 1 and 3 are linearly independent, so a basis for the column space is formed by the vectors $\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$ and $\begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix}$: namely, columns 1 and 3 *of the original matrix.*

   Remark Other answers are possible. For instance, columns 1 and 4 of the original matrix would do as well, as would columns 3 and 4 of the original matrix.

   An alternative approach (not discussed in the textbook) would be to "column reduce" the original matrix. Since you are not used to performing column operations, you could row reduce the transpose matrix and then transpose the result. This method produces a basis for the column space consisting of the two vectors $\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$ and $\begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$.

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2. In the space $C[0,1]$ of continuous functions on the interval $[0,1]$, the functions e^x and e^{-x} span a two-dimensional subspace. One basis for this subspace, call it the E basis, is $[e^x, e^{-x}]$. Another basis, call it the H basis, is $[\cosh(x), \sinh(x)]$, where the so-called hyperbolic functions are defined as follows:

$$\cosh(x) = \frac{e^x + e^{-x}}{2} \qquad \text{and} \qquad \sinh(x) = \frac{e^x - e^{-x}}{2}.$$

Find the transition matrix A from the H basis to the E basis. In other words, find the 2×2 matrix A with the property that if

$$\begin{aligned} f(x) &= a\cosh(x) + b\sinh(x) \\ &= ce^x + de^{-x}, \end{aligned}$$

then $A\begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} c \\ d \end{pmatrix}$.

Solution. Setting $\begin{pmatrix} a \\ b \end{pmatrix}$ equal to $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ picks off the first column of the matrix A. Since

$$1 \cdot \cosh(x) + 0 \cdot \sinh(x) = \tfrac{1}{2}e^x + \tfrac{1}{2}e^{-x},$$

the first column of A equals $\begin{pmatrix} 1/2 \\ 1/2 \end{pmatrix}$. Similarly, setting $\begin{pmatrix} a \\ b \end{pmatrix}$ equal to $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$ picks off the second column of the matrix A. Since

$$0 \cdot \cosh(x) + 1 \cdot \sinh(x) = \tfrac{1}{2}e^x - \tfrac{1}{2}e^{-x},$$

the second column of A equals $\begin{pmatrix} 1/2 \\ -1/2 \end{pmatrix}$. Thus

$$A = \begin{pmatrix} 1/2 & 1/2 \\ 1/2 & -1/2 \end{pmatrix}.$$

Remark The problem does not ask for A^{-1}, which is $\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$.