Linear Algebra

Instructions Please write your name in the upper right-hand corner of the page. Use complete sentences, along with any necessary supporting calculations, to answer the following questions.

1. Suppose $A = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$ and $B = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$. The matrix A represents the linear operator

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \mapsto \begin{pmatrix} -x_1 \\ x_2 \end{pmatrix}$$

on R^2 with respect to the standard basis $\left[\begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix}\right]$, and the matrix B represents the same operator with respect to the nonstandard basis $\left[\begin{pmatrix} 1 \\ 1 \end{pmatrix}, \begin{pmatrix} -1 \\ 1 \end{pmatrix}\right]$. Find a matrix S such that $S^{-1}AS = B$.

Solution. The required matrix S is the transition matrix from the nonstandard basis to the standard basis. The columns of this transition matrix are the nonstandard basis vectors, so $S = \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}$.

Remark The answer is not unique. Since the matrix B corresponds to interchanging the two nonstandard basis vectors, the order of these basis vectors does not matter in this problem. Hence interchanging the two columns of the matrix S gives another correct answer: namely, $\begin{pmatrix} -1 & 1 \\ 1 & 1 \end{pmatrix}$.

More generally, the matrix B represents our linear operator with respect to any basis of the form $\left[\begin{pmatrix} a \\ b \end{pmatrix}, \begin{pmatrix} -a \\ b \end{pmatrix}\right]$, where a and b are arbitrary nonzero numbers. Therefore S could be any matrix of the form $\begin{pmatrix} a & -a \\ b & b \end{pmatrix}$, where $a \neq 0$ and $b \neq 0$.

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2. In the space R^3 equipped with its standard scalar product, find the vector projection of the vector $\begin{pmatrix}2\\4\\3\end{pmatrix}$ onto the vector $\begin{pmatrix}1\\1\\1\end{pmatrix}$.
[This is exercise 3(c) on page 224 of the textbook.]

Solution. The scalar projection is the scalar product of $\begin{pmatrix}2\\4\\3\end{pmatrix}$ with the unit vector $\dfrac{1}{\sqrt{3}}\begin{pmatrix}1\\1\\1\end{pmatrix}$: namely, $\dfrac{9}{\sqrt{3}}$. The required vector projection is the scalar projection times the unit vector $\dfrac{1}{\sqrt{3}}\begin{pmatrix}1\\1\\1\end{pmatrix}$: namely, $\begin{pmatrix}3\\3\\3\end{pmatrix}$.