Topics in Applied Mathematics I

Each of the 10 problems counts 10 points.
Show your work to obtain full credit.

1. Determine the null space of the matrix $\begin{pmatrix} 0 & 1 & 2 \\ 3 & 4 & 5 \\ 6 & 7 & 8 \end{pmatrix}$.

   Row reduce: $\begin{pmatrix} 0 & 1 & 2 \\ 3 & 4 & 5 \\ 6 & 7 & 8 \end{pmatrix} \to \begin{pmatrix} 0 & 1 & 2 \\ 3 & 4 & 5 \\ 0 & -1 & -2 \end{pmatrix} \to \begin{pmatrix} 0 & 1 & 2 \\ 3 & 0 & -3 \\ 0 & 0 & 0 \end{pmatrix}$.

   From this reduced form, you can read off that the null space consists of all scalar multiples of the vector $(1, -2, 1)$.

2. Either give an example of a matrix representing a linear transformation $f\colon \mathbb{R}^3 \to \mathbb{R}^3$ such that the image is the plane spanned by the vectors $(1, 2, 3)$ and $(4, 5, 6)$ or else explain why no such transformation exists.

   The answer is not unique. The simplest solution is to put the given two vectors as columns of the matrix, and to make the third column linearly dependent on those two columns. Some popular answers are $\begin{pmatrix} 1 & 4 & 0 \\ 2 & 5 & 0 \\ 3 & 6 & 0 \end{pmatrix}$ and $\begin{pmatrix} 1 & 4 & 5 \\ 2 & 5 & 7 \\ 3 & 6 & 9 \end{pmatrix}$ (the third column is the sum of the first two) and $\begin{pmatrix} 1 & 4 & 2 \\ 2 & 5 & 4 \\ 3 & 6 & 6 \end{pmatrix}$ (the third column is twice the first column).

3. On the vector space $\mathcal{P}_2$ of polynomials of degree less than or equal to 2, the operation of differentiation is a linear transformation. What is the dimension of the null space? What is the dimension of the image? Explain why.

   The null space consists of all functions with derivative equal to 0, namely all constant functions. This is a space of dimension 1 (with basis $\{1\}$). The image consists of all polynomials of degree less than or equal to 1, a space of dimension 2. (A basis for the image is $\{1, x\}$.) The sum of these dimensions is 3, which is the dimension of the domain $\mathcal{P}_2$.

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4. Suppose $A = \begin{pmatrix} 2 & t & 0 \\ 0 & 3 & 1 \\ 0 & 0 & 2 \end{pmatrix}$. For which value(s) of t, if any, does $\mathbb{R}^3$ have a basis consisting of eigenvectors of the matrix A? Explain why.

   The matrix A is upper triangular, so the eigenvalues are the entries on the diagonal: 2, 3, and 2.

   Eigenvectors corresponding to eigenvalue 3 form the null space of the matrix $A - 3I = \begin{pmatrix} -1 & t & 0 \\ 0 & 0 & 1 \\ 0 & 0 & -1 \end{pmatrix}$. The eigenvectors are all scalar multiples of the vector $(t, 1, 0)$.

   Eigenvectors corresponding to eigenvalue 2 form the null space of the matrix $A - 2I = \begin{pmatrix} 0 & t & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{pmatrix}$. If $t \neq 0$, this matrix row reduces to $\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$, and the eigenvectors are all scalar multiples of the vector $(1, 0, 0)$. In this case, there are two linearly independent eigenvectors (one for eigenvalue 3 and one for eigenvalue 2), so $\mathbb{R}^3$ does not have a basis consisting of eigenvectors of the matrix A: the eigenvectors span a 2-dimensional subspace of $\mathbb{R}^3$.

   If $t = 0$, however, then the matrix $A - 2I$ has two rows of zeroes, and the eigenvectors corresponding to eigenvalue 2 are all linear combinations of the vectors $(1, 0, 0)$ and $(0, 1, -1)$. Together with the vector $(0, 1, 0)$ (the eigenvector corresponding to eigenvalue 3 when $t = 0$) these vectors do form a basis for $\mathbb{R}^3$.

5. The formula $\langle (x_1, x_2), (y_1, y_2) \rangle = 2x_1y_1 + x_1y_2 + x_2y_1 + x_2y_2$ defines a non-standard inner product on $\mathbb{R}^2$.

   (a) Verify that the vector $(\frac{3}{5}, \frac{1}{5})$ has norm (length) equal to 1 with respect to this non-standard inner product.

   (b) Find a second vector that, together with the vector $(\frac{3}{5}, \frac{1}{5})$, forms an orthonormal basis of $\mathbb{R}^2$ with respect to the indicated non-standard inner product.

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For this non-standard inner product, $\|(x_1, x_2)\|^2 = 2x_1^2 + 2x_1x_2 + x_2^2$, so $\|(\frac{3}{5}, \frac{1}{5})\|^2 = 2(\frac{3}{5})^2 + 2(\frac{3}{5})(\frac{1}{5}) + (\frac{1}{5})^2 = \frac{18}{25} + \frac{6}{25} + \frac{1}{25} = 1$.
To get a second vector that is orthogonal to $(\frac{3}{5}, \frac{1}{5})$, start with any vector, say $(1, 0)$, and subtract its projection on $(\frac{3}{5}, \frac{1}{5})$ with respect to the inner product. Since the inner product $\langle(1, 0), (\frac{3}{5}, \frac{1}{5})\rangle = \frac{6}{5}+\frac{1}{5}+0+0 = \frac{7}{5}$, the vector $(1, 0)-\frac{7}{5}(\frac{3}{5}, \frac{1}{5}) = (\frac{4}{25}, -\frac{7}{25})$ is orthogonal to $(\frac{3}{5}, \frac{1}{5})$. Dividing $(\frac{4}{25}, -\frac{7}{25})$ by its norm gives the normalized vector $(\frac{4}{5}, -\frac{7}{5})$.

6. Suppose $f(t) = (\cos(t), \cos(2t), \cos(3t))$ represents a curve in $\mathbb{R}^3$ given in parametric form. Find an equation for the line tangent to the curve at the point where $t = \pi/2$.

   The point on the curve is $(\cos(\pi/2), \cos(\pi), \cos(3\pi/2)) = (0, -1, 0)$. The tangent vector is $f'(t) = (-\sin(t), -2\sin(2t), -3\sin(3t))$. At the point where $t = \pi/2$, the tangent vector is $f'(\pi/2) = (-1, 0, 3)$. A parametric form for the tangent line is $(0, -1, 0) + s(-1, 0, 3)$.

7. Suppose $g(u, v) = (u, v, u^2 + v^2)$ represents a surface in $\mathbb{R}^3$ given in parametric form. (This surface is a paraboloid.) Find an equation for the plane tangent to the surface at the point where $u = 1$ and $v = 2$.

   One tangent vector is $\frac{\partial g}{\partial u}\big|_{(1,2)} = (1, 0, 2u)\big|_{(1,2)} = (1, 0, 2)$. A second tangent vector is $\frac{\partial g}{\partial v}\big|_{(1,2)} = (0, 1, 2v)\big|_{(1,2)} = (0, 1, 4)$. A vector normal to the plane is the cross product $(-2, -4, 1)$. A point on the plane is $g(1, 2) = (1, 2, 5)$. An equation for the plane is $-2(x-1)-4(y-2)+(z-5) = 0$ or $-2x - 4y + z = -5$ or $2x + 4y - z = 5$.

8. Suppose $f(x, y, z) = x^2 - yz$. Find a unit vector $\vec{u}$ such that the directional derivative of the function f in the direction $\vec{u}$ at the point $(1, 2, 3)$ is 0.

   The gradient of f is $(2x, -z, -y)$, so $\nabla f(1, 2, 3) = (2, -3, -2)$. We want a unit vector orthogonal to this vector. There is a two-dimensional subspace of vectors in $\mathbb{R}^3$ orthogonal to $(2, -3, -2)$, so the answer is not unique. Two popular choices are $(1, 0, 1)/\sqrt{2}$ and $(3, 2, 0)/\sqrt{13}$.

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9. Suppose that u, v, and w are functions of x and y, while x and y are functions of r, s, and t. The following information is given.

   - Both x and y are equal to 0 when $r = s = t = 0$.
   - At the point where $x = y = 0$, the partial derivatives of u, v, and w with respect to x and y have these values: $\dfrac{\partial u}{\partial x} = \dfrac{\partial u}{\partial y} = 1$, $\dfrac{\partial v}{\partial x} = \dfrac{\partial v}{\partial y} = 2$, $\dfrac{\partial w}{\partial x} = \dfrac{\partial w}{\partial y} = 3$.
   - At the point where $r = s = t = 0$, the partial derivatives of x and y with respect to r, s, and t have these values: $\dfrac{\partial x}{\partial r} = \dfrac{\partial y}{\partial r} = 4$, $\dfrac{\partial x}{\partial s} = \dfrac{\partial y}{\partial s} = 5$, $\dfrac{\partial x}{\partial t} = \dfrac{\partial y}{\partial t} = 6$.

   Compute the value of $\dfrac{\partial v}{\partial t}$ at the point where $r = s = t = 0$.

   According to the chain rule, $\frac{\partial v}{\partial t} = \frac{\partial v}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial v}{\partial y}\frac{\partial y}{\partial t} = 2 \times 6 + 2 \times 6 = 24$.

10. Evaluate the double integral $\int_0^1 \int_0^{\sqrt{1-x^2}} 2e^{-x^2-y^2}\,dy\,dx$ by transforming the integral to polar coordinates. (The integration region is one-quarter of a circular disk.)

    The integral transforms to $\int_0^{\pi/2} \int_0^1 2e^{-r^2} r\,dr\,d\theta = \int_0^{\pi/2} \left[-e^{-r^2}\right]_0^1 d\theta = \frac{\pi}{2}(-e^{-1}+1) = \frac{\pi(e-1)}{2e}$.

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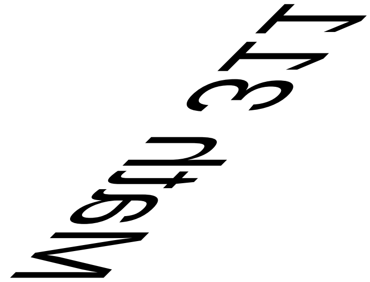
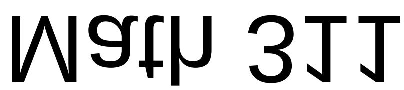
Extra credit (up to 5 points): I used a linear transformation of $\mathbb{R}^2$ to change

Math 311 into 

What 2×2 matrix did I use to define the transformation?

The matrix $\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ represents a reflection in the first coordinate axis, which transforms the input into . The matrix $\begin{pmatrix} 1/\sqrt{2} & -1/\sqrt{2} \\ 1/\sqrt{2} & 1/\sqrt{2} \end{pmatrix}$ represents a counterclockwise rotation by 45°. The composition of those two transformations, represented by the product matrix $\begin{pmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ 1/\sqrt{2} & -1/\sqrt{2} \end{pmatrix}$, transforms the input into .

This is not yet the right transformation, because we want the letters to be stroked parallel to the first coordinate axis, not perpendicular to the 45° line. The effect can be achieved by reflecting, slanting the letters with the transformation $\begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}$, and then rotating. The final transformation is the matrix product $\begin{pmatrix} 1/\sqrt{2} & -1/\sqrt{2} \\ 1/\sqrt{2} & 1/\sqrt{2} \end{pmatrix} \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = \begin{pmatrix} 1/\sqrt{2} & \sqrt{2} \\ 1/\sqrt{2} & 0 \end{pmatrix}$.