Math 311-102

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Overview

Yesterday: systems of equations and row reduction.

For example, the system $\begin{pmatrix} 1 & 2 & 1 \\ 4 & 3 & -11 \\ 5 & -1 & -28 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 12 \\ 13 \\ -17 \end{pmatrix}$

has the same solutions as the reduced system

$$\begin{pmatrix} 1 & 0 & -5 \\ 0 & 1 & 3 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -2 \\ 7 \\ 0 \end{pmatrix}.$$

Today: matrix algebra (sums, products, inverses). For example, find a matrix M such that the matrix product

$$M \begin{pmatrix} 1 & 2 & 1 \\ 4 & 3 & -11 \\ 5 & -1 & -28 \end{pmatrix} = \begin{pmatrix} 1 & 0 & -5 \\ 0 & 1 & 3 \\ 0 & 0 & 0 \end{pmatrix}.$$

Componentwise operations

Sums and differences of matrices are computed componentwise.

Example. $\begin{pmatrix} 10 & 20 \\ 30 & 40 \end{pmatrix} - \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} = \begin{pmatrix} 9 & 18 \\ 27 & 36 \end{pmatrix}.$

Multiplication of a matrix by a scalar is computed componentwise.

Example. $10 \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} = \begin{pmatrix} 10 & 20 \\ 30 & 40 \end{pmatrix}.$

The product of a matrix with a matrix, however, is *not* computed componentwise.

Matrix multiplication

The product of a matrix with a column vector is a new vector obtained by taking dot products with the rows of the matrix:

$$\begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix} \begin{pmatrix} 1 \\ 10 \\ 100 \end{pmatrix} = \begin{pmatrix} 321 \\ 654 \\ 987 \end{pmatrix}.$$

The product of a matrix with another matrix is computed similarly by letting the first matrix act on each column of the second matrix: $\begin{pmatrix} 1 & 0 & 2 \\ 0 & 3 & 4 \\ 5 & 6 & 0 \end{pmatrix} \begin{pmatrix} 1 & 5 \\ 10 & 50 \\ 100 & 500 \end{pmatrix} = \begin{pmatrix} 201 & 1005 \\ 430 & 2150 \\ 65 & 325 \end{pmatrix}.$

The product of matrices makes sense only if the rows of the first matrix have the same number of entries as the columns of the second matrix.

Multiplication is not commutative!

Example.
$\begin{pmatrix} 1 & 0 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ -1 & -1 \end{pmatrix}$, but
$\begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$.

The order of multiplication matters: $AB \neq BA$.

On the other hand, matrix multiplication is associative: $A(BC) = (AB)C$.

Notice in the above example that the product of two non-zero matrices can be the zero matrix. In particular, only certain matrices have multiplicative inverses.

Identity and inverses

For 3×3 matrices, the *identity matrix* $I = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ has the property that $IA = A$ and $AI = A$ for every 3×3 matrix A (and similarly for square matrices of other sizes).

Two matrices A and B are *inverses* if $AB = I$ and $BA = I$.

For 2×2 matrices, there is a simple rule for the inverse of a matrix:

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}^{-1} = \frac{1}{ad - bc} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}.$$

This inverse exists if and only if the *determinant* $(ad - bc) \neq 0$.

Computing inverses in general

Method: write the matrix next to an identity matrix and do row operations on both matrices simultaneously. When the initial matrix has been turned into the identity matrix, the identity matrix will have been turned into the inverse matrix. Example:

$$\begin{pmatrix} 1 & 0 & -1 \\ 2 & 1 & 0 \\ -2 & 0 & 1 \end{pmatrix} \left| \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \right. \qquad \text{row reduce} \rightarrow$$

$$\begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & 2 \\ 0 & 0 & -1 \end{pmatrix} \left| \begin{pmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 2 & 0 & 1 \end{pmatrix} \right. \rightarrow \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \left| \begin{pmatrix} -1 & 0 & -1 \\ 2 & 1 & 2 \\ -2 & 0 & -1 \end{pmatrix} \right.$$

Conclusion: $\begin{pmatrix} 1 & 0 & -1 \\ 2 & 1 & 0 \\ -2 & 0 & 1 \end{pmatrix}^{-1} = \begin{pmatrix} -1 & 0 & -1 \\ 2 & 1 & 2 \\ -2 & 0 & -1 \end{pmatrix}$.