Math 311-102

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Overview

A square matrix has an inverse if and only if the *determinant* of the matrix is different from 0.

We saw last time how to compute the inverse matrix by row reduction.

Plan for today
1. How to compute determinants
2. How to use determinants
   a) to compute inverse matrices
   b) to solve systems of equations

Computing determinants

For 2×2 matrices, $\det \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$.

For $n \times n$ matrices, the determinant can be defined recursively in terms of *cofactors*.

The cofactor of an element of a matrix is plus or minus the determinant of the submatrix that remains when the row and column of the element are crossed off.
The sign is chosen according to the pattern

$$\begin{pmatrix} + & - & + & - & \cdots \\ - & + & - & + & \cdots \\ + & - & + & - & \cdots \\ - & + & - & + & \cdots \\ \vdots & \vdots & \vdots & \vdots & \end{pmatrix}.$$

Cofactors

Example.

In the matrix $\begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix}$, the cofactor of the element 4 is

$-\det \begin{pmatrix} 2 & 3 \\ 8 & 9 \end{pmatrix} = -(18 - 24) = 6.$

The cofactor of the element 3 is $+\det \begin{pmatrix} 4 & 5 \\ 7 & 8 \end{pmatrix} = 32 - 35 = -3.$

Computing determinants (continued)

Recursive rule for the determinant of a matrix:
Pick any one row or column, multiply each element in that row or column by its cofactor, and add the results.

Example. Expanding $\det\begin{pmatrix}1 & 0 & 2\\ 0 & 3 & 5\\ -1 & 4 & 0\end{pmatrix}$ across the bottom row

gives $-1\det\begin{pmatrix}0 & 2\\ 3 & 5\end{pmatrix} - 4\det\begin{pmatrix}1 & 2\\ 0 & 5\end{pmatrix} + 0\det\begin{pmatrix}1 & 0\\ 0 & 3\end{pmatrix}$
$= -1(0-6) - 4(5-0) + 0 = -14.$

Expanding on the third column gives
$2\det\begin{pmatrix}0 & 3\\ -1 & 4\end{pmatrix} - 5\det\begin{pmatrix}1 & 0\\ -1 & 4\end{pmatrix} + 0\det\begin{pmatrix}1 & 0\\ 0 & 3\end{pmatrix}$
$= 2(3) - 5(4) + 0 = -14.$

Inverse matrices and determinants

Formula for the inverse of a matrix:

Replace each element by its cofactor (remember the $\pm$ sign), transpose the resulting matrix, and divide by the determinant of the original matrix.

Example. If $A = \begin{pmatrix}1 & 2 & 3\\ 0 & 4 & 5\\ 0 & 0 & 6\end{pmatrix}$, then $\det A = 24$, and

$$A^{-1} = \frac{1}{24}\begin{pmatrix} \left|\begin{pmatrix}4 & 5\\ 0 & 6\end{pmatrix}\right| & -\left|\begin{pmatrix}2 & 3\\ 0 & 6\end{pmatrix}\right| & \left|\begin{pmatrix}2 & 3\\ 4 & 5\end{pmatrix}\right| \\ -\left|\begin{pmatrix}0 & 5\\ 0 & 6\end{pmatrix}\right| & \left|\begin{pmatrix}1 & 3\\ 0 & 6\end{pmatrix}\right| & -\left|\begin{pmatrix}1 & 3\\ 0 & 5\end{pmatrix}\right| \\ \left|\begin{pmatrix}0 & 4\\ 0 & 0\end{pmatrix}\right| & -\left|\begin{pmatrix}1 & 2\\ 0 & 0\end{pmatrix}\right| & \left|\begin{pmatrix}1 & 2\\ 0 & 4\end{pmatrix}\right| \end{pmatrix}.$$

Example continued

So $A = \begin{pmatrix}1 & 2 & 3\\ 0 & 4 & 5\\ 0 & 0 & 6\end{pmatrix}$ and $A^{-1} = \begin{pmatrix}1 & -\frac{1}{2} & -\frac{1}{12}\\ 0 & \frac{1}{4} & -\frac{5}{24}\\ 0 & 0 & \frac{1}{6}\end{pmatrix}$.

You can check the answer by multiplying A times A^{-1} to see if you get the identity matrix.

Notice that the inverse of an *upper triangular* matrix is again upper triangular.

Properties of determinants

What happens to the determinant under elementary row operations?

1. Adding a multiple of a row to another row leaves the determinant unchanged. **Example**. $R2 \to R2 + 2R1$:
$\begin{vmatrix}1 & 10\\ -2 & 3\end{vmatrix} = \begin{vmatrix}1 & 10\\ 0 & 23\end{vmatrix}.$

2. Multiplying a row by a scalar multiplies the determinant by that scalar.
Example. $\begin{vmatrix}5 & 50\\ -2 & 3\end{vmatrix} = 5\begin{vmatrix}1 & 10\\ -2 & 3\end{vmatrix} = 115.$

3. Interchanging two rows changes the sign of the determinant.
Example. $\begin{vmatrix}-2 & 3\\ 1 & 10\end{vmatrix} = -\begin{vmatrix}1 & 10\\ -2 & 3\end{vmatrix}.$

Example: problem 6, page 98

$$\begin{vmatrix}1&1&1&1\\1&2&4&8\\1&3&9&27\\1&4&16&64\end{vmatrix} = \begin{vmatrix}1&1&1&1\\0&1&3&7\\0&2&8&26\\0&3&15&63\end{vmatrix}$$ by subtracting the first row from the other rows

$$= \begin{vmatrix}1&3&7\\2&8&26\\3&15&63\end{vmatrix}$$ by expanding on the first column

$$= \begin{vmatrix}1&3&7\\0&2&12\\0&6&42\end{vmatrix}$$ via $R2 \to R2 - 2R1$ and $R3 \to R3 - 3R1$

$$= \begin{vmatrix}2&12\\6&42\end{vmatrix} = 2\begin{vmatrix}1&6\\6&42\end{vmatrix} = 12\begin{vmatrix}1&6\\1&7\end{vmatrix} = 12.$$

Cramer's rule

Formula for solving a system of equations $A\vec{x} = \vec{b}$:
$x_1 = \frac{1}{\det A}\det($matrix A with its first column replaced by $\vec{b})$
and similarly for $x_2, x_3, \ldots$.

Example. $\begin{pmatrix}1&0&4\\0&2&3\\5&1&0\end{pmatrix}\begin{pmatrix}x_1\\x_2\\x_3\end{pmatrix} = \begin{pmatrix}10\\20\\30\end{pmatrix}$

$$x_1 = \frac{\begin{vmatrix}10&0&4\\20&2&3\\30&1&0\end{vmatrix}}{\begin{vmatrix}1&0&4\\0&2&3\\5&1&0\end{vmatrix}} \qquad x_2 = \frac{\begin{vmatrix}1&10&4\\0&20&3\\5&30&0\end{vmatrix}}{\begin{vmatrix}1&0&4\\0&2&3\\5&1&0\end{vmatrix}} \qquad x_3 = \frac{\begin{vmatrix}1&0&10\\0&2&20\\5&1&30\end{vmatrix}}{\begin{vmatrix}1&0&4\\0&2&3\\5&1&0\end{vmatrix}}$$

$x_1 = 190/43$ $\qquad x_2 = 340/43$ $\qquad x_3 = 60/43$