Math 311-102

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Linear functions

A function f whose domain is n-dimensional space $\mathbb{R}^n$ and whose range is $\mathbb{R}^m$ is *linear* if it respects linear combinations: namely, $f(a\vec{x} + b\vec{y}) = af(\vec{x}) + bf(\vec{y})$ for all vectors $\vec{x}$ and $\vec{y}$ and all scalars a and b.

Examples. Are the following functions linear?

(i) $f(x_1, x_2) = x_1$ (domain $\mathbb{R}^2$, range $\mathbb{R}^1$)
Yes; this function is *projection* onto the first coordinate axis.

(ii) $f(x_1, x_2) = (-x_2, x_1)$ (domain $\mathbb{R}^2$, range $\mathbb{R}^2$)
Yes; this function is *rotation* by 90° counterclockwise.

(iii) $f(x) = (x, 3x, x^2)$ (domain $\mathbb{R}$, range $\mathbb{R}^3$)
No, because $(x+y)^2 \neq x^2 + y^2$.

(iv) $f(x) = 5x + 7$ (domain $\mathbb{R}$, range $\mathbb{R}$)
No! $f(x+y) = 5x + 5y + 7$, but $f(x) + f(y) = 5x + 5y + 14$.
A function like this (the sum of a linear function and a constant) is sometimes called *affine* or *affine linear*.

Matrix representation

Example. Suppose $f : \mathbb{R}^2 \to \mathbb{R}^2$ is linear, $f(\vec{e}_1) = \begin{pmatrix} 3 \\ 4 \end{pmatrix}$, and $f(\vec{e}_2) = \begin{pmatrix} 10 \\ 20 \end{pmatrix}$. Find a formula for $f(\vec{x})$.

Solution. Since $\vec{x} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = x_1 \begin{pmatrix} 1 \\ 0 \end{pmatrix} + x_2 \begin{pmatrix} 0 \\ 1 \end{pmatrix}$, linearity says that $f(\vec{x}) = x_1 f(\vec{e}_1) + x_2 f(\vec{e}_2) = x_1 \begin{pmatrix} 3 \\ 4 \end{pmatrix} + x_2 \begin{pmatrix} 10 \\ 20 \end{pmatrix}$

$$= \begin{pmatrix} 3x_1 + 10x_2 \\ 4x_1 + 20x_2 \end{pmatrix} = \begin{pmatrix} 3 & 10 \\ 4 & 20 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}.$$

Every linear function is represented by a matrix whose columns are the images of the standard basis vectors.

Example: rotation

Find the matrix representation of counterclockwise rotation in the plane by angle 30°.

Solution. The image of the vector $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ is the vector $\begin{pmatrix} \cos(30^\circ) \\ \sin(30^\circ) \end{pmatrix} = \begin{pmatrix} \sqrt{3}/2 \\ 1/2 \end{pmatrix}$, and the image of the vector $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$ is $\begin{pmatrix} -1/2 \\ \sqrt{3}/2 \end{pmatrix}$, so the matrix representing the rotation is $\begin{pmatrix} \sqrt{3}/2 & -1/2 \\ 1/2 & \sqrt{3}/2 \end{pmatrix}$.

In general, the matrix representing a counterclockwise rotation in $\mathbb{R}^2$ by angle θ is $\begin{pmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{pmatrix}$.

One-to-one functions

A function f is *one-to-one* (or *injective*) if no two distinct points of the domain have equal images: $f(\vec{x}) \neq f(\vec{y})$ when $\vec{x} \neq \vec{y}$.

To show that a *linear* function is one-to-one, it is enough to check that $f(\vec{x}) = \vec{0}$ only when $\vec{x} = \vec{0}$.

Equivalently, $f(\vec{x}) = A\vec{x}$ is a one-to-one function if the columns of the matrix A are linearly independent.

Example. If $f(\vec{x}) = \begin{pmatrix} 1 & 2 & 1 \\ 2 & 1 & 5 \\ 3 & 4 & 5 \\ 4 & 7 & 5 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$, is f one-to-one?

Answer. No. The reduced form of the matrix is $\begin{pmatrix} 1 & 0 & 3 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$,

so $3C1 - C2 - C3 = 0$; thus $f(3, -1, -1) = \vec{0}$.