Math 311-102

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Vector spaces

A *vector space* is a set of mathematical objects with an addition law that is commutative and associative and a multiplication by scalars that satisfies the distributive law.

Examples.
(i) $\mathbb{R}^n$ (the standard model of a vector space)
(ii) $\mathbb{R}^\infty$, the space of unending sequences $(x_1, x_2, \ldots)$
(iii) $\mathcal{P}_n$, the space of polynomials of degree $\leq n$
(iv) $C[0, 1]$, the space of continuous functions on the closed interval $[0, 1]$
(v) $\mathcal{M}_{n,m}$, the space of matrices with n rows and m columns

Non-examples.
(i) polynomials with constant term 1 (not closed under $+$)
(ii) real numbers > 0 (not closed under multiplication by -1)
(iii) matrices with determinant equal to 0 (not closed under $+$)

Subspaces

A typical way to get a new vector space from an old one is to take a subset that is closed under addition and under multiplication by scalars. This is called a *subspace*.

Examples.
(i) The set of vectors in $\mathbb{R}^3$ perpendicular to the vector $(1, 2, 3)$ is a subspace of $\mathbb{R}^3$: namely, the plane $x + 2y + 3z = 0$.
(ii) Every plane passing through the origin is a subspace of $\mathbb{R}^3$. So is every line passing through the origin.
(iii) If $\mathcal{M}_{2,2}$ is the vector space of 2×2 matrices, then the set of *symmetric* 2×2 matrices is a subspace.
(iv) If $\mathcal{P}$ is the vector space of all polynomials, then the set of polynomials $p(x)$ such that $\int_0^1 p(x)\,dx = 0$ is a subspace.
(v) If $C[0, 1]$ is the vector space of continuous functions on the interval $[0, 1]$, then the set of continuous functions f such that $f(1) = 0$ is a subspace.

Linear transformations

Yesterday we called a function f with domain $\mathbb{R}^n$ and range $\mathbb{R}^m$ *linear* if $f(\vec{x} + \vec{y}) = f(\vec{x}) + f(\vec{y})$ and $f(a\vec{x}) = af(\vec{x})$ for all vectors $\vec{x}$ and $\vec{y}$ and all scalars a.

In general, a function between vector spaces is similarly called a *linear transformation* (or *linear operator*) if it respects sums and multiplication by scalars.

Examples.
(i) Differentiation is a linear operator on the space $\mathcal{P}$ of polynomials because $(p(x) + q(x))' = p'(x) + q'(x)$ and $(ap(x))' = ap'(x)$.
(ii) Another linear operator on $\mathcal{P}$ is $p(x) \mapsto \int_0^x p(t)\,dt$ because $\int_0^x (p(t) + q(t))\,dt = \int_0^x p(t)\,dt + \int_0^x q(t)\,dt$ and $\int_0^x ap(t)\,dt = a\int_0^x p(t)\,dt$.

Non-example. $p(x) \mapsto p(x)p'(x)$ is not linear.

Image and inverse

If $f(\vec{x}) = A\vec{x}$, then the *image* of f consists of all linear combinations of the columns of the matrix A.

Example. If $f(\vec{x}) = \begin{pmatrix} 1 & 2 \\ 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$, then the image of f is a plane in $\mathbb{R}^3$: namely, the plane $x - 2y + z = 0$.

The image of a linear transformation between vector spaces is always a subspace of the range.

When f is one-to-one, the *inverse* f^{-1} satisfies $f^{-1}(f(\vec{x})) = \vec{x}$ for every vector $\vec{x}$ in the domain. The inverse is again a linear transformation.

In the preceding example, f^{-1} can be realized by the matrix
$\begin{pmatrix} 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}$ since $\begin{pmatrix} 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 2 \\ 0 & 1 \\ -1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$.